The Fibonacci Project

DISSEMINATING INQUIRY-BASED SCIENCE AND MATHEMATICS EDUCATION IN EUROPE

VOLKER ULM (ED.)

Inquiry-Based Mathematics Education for Gifted Children in Primary School



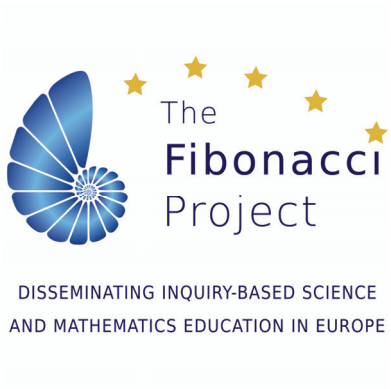
WITH THE SUPPORT OF

EUROPEAN COMMISION

SEVENTH FRAMEWORK PROGRAMME

Volker Ulm (Ed.)

Inquiry-Based Mathematics Education for Gifted Children in Primary School



The CD-ROM provides:

- worksheets for the pupils as Microsoft® Word® documents and PDF files, and
- background information as PDF files.

ISBN 978-3-00-033657-7

© 2011 University of Augsburg

Translation from German to English: Kim Freundlich, Augsburg
Natalie Selinski, Augsburg
Book cover: www.lezard-graphique.net
Illustrations: Dietmar Griese, Laatzen
Print: manageyourmedia, Augsburg

This book is originally published in German: "Ulm, V. (Hrsg.): Mathematische Begabungen fördern, Cornelsen Verlag Scriptor, Berlin 2010" (ISBN 978-3-589-05157-1)
The English version may not be used for commercial purposes. The contents can be translated into any language except for German and may be used for non-commercial purposes. Translating this book or any portion of the content of this book into German is not permitted. This information must be included in every other translation of this work.

The Fibonacci project deals with disseminating inquiry-based science and mathematics education in Europe. Within Europe, strategies in order to develop methods of teaching mathematics and science and concepts for teachers' professional development are designed, tested, evaluated and implemented. For further information on the Fibonacci project visit: http://fibonacci-project.eu

The Fibonacci Project has been funded with support from the European Commission in the European Union's Seventh Framework Programme. This communication reflects the views only of the authors. The Commission cannot be held responsible for any use which may be made of the information contained therein.

Contents

Support Mathematically Gifted Children
(*Volker Ulm*) 5

Arithmetic

1. Exploring Magic Squares
(*Doris Bocka*) 10

2. Windows on the Hundreds Chart
(*Petra Ihn-Huber*) 22

3. Exploring the System
(*Matthias Brandl, Birgit Brandl*) 30

4. The Postman and Fibonacci
(*Doris Bocka*) 36

5. Experimenting with Number Cards
(*Karsten Weigl*) 38

6. A World Where People Have Only Six Fingers
(*Renate Motzer*) 44

Practical Applications

7. Arranging High-Speed Trains
(*Karsten Weigl*) 50

8. Placing Marbles into Bags
(*Jutta Schoft, Tanja Sinning, Annette Stix-Pöhner*) 52

9. Secret Languages
(*Volker Ulm*) 56

10. Fermi Problems
(*Barbara Adleff*) 60



Geometry

11. Paper Formats and Origami
(*Ingrid Weigand*) 62

12. Geometry through Paper Folding: The Origami Fox
(*Ingrid Weigand*) 66

13. Brain-Teasers with Pentacubes
(*Birgit Brandl, Matthias Brandl*) 70

14. Measuring Our School
(*Tanja Sinning, Annette Stix-Pöhner*) 80

15. Fencing In a Sheep Pasture
(*Karsten Weigl*) 86

16. Experimenting with Dice
(*Doris Bocka*) 88

17. Pentominoes
(*Ingrid Weigand*) 92

V. Ulm

1 The Purpose of this Book

This book has been written for primary school teachers who would like to challenge mathematically gifted pupils or pupils with a particular interest in mathematics. Given the heterogeneity in schools, it is necessary to consider the pupils' different learning needs and abilities. On the one hand, mathematically weaker pupils need more support and appropriate exercises. On the other hand, mathematically gifted pupils should be challenged in order to develop their skills and interests. Good mathematics lessons are aimed at educating *all* pupils – even if this can only occur at different levels due to the pupils' different levels of mathematical proficiency.

If a teacher would like to support mathematically gifted pupils, he or she will need teaching and learning materials which are based on the curriculum but also develop a deeper understanding of mathematics. This book satisfies these needs by offering various worksheets with didactic commentary. The worksheets can be used in various educational settings:

- The topics can be worked on regularly in school together with all children, and the tasks can be assigned according to the students' levels of mathematical proficiency.
- The exercises can be used as additional tasks for very interested and gifted students to work on either at school or at home.
- These topics are also suited for study groups outside of the classroom setting, where students deal with mathematical problems.

The tasks should encourage the pupils to experiment and learn through inquiry-based learning. Therefore, the children take on the role of explorers who find and develop mathematics in various situations. As such, the pupils should:

- Work on the topics independently and cooperatively, and find their own ways to discover the mathematics on which the exercises are based;
- Discuss their ideas and possible solutions in groups;
- Write down their ideas and describe their results; and
- Present and explain their ideas and results to the entire class.

Because the learning materials offered in this book are directed particularly towards mathematically gifted primary school pupils, the terms "mathematical thinking", "mathematical giftedness", "mathematical abilities" and "mathematical performance", each of which is significant to the conceptual foundation of this book, are explained and distinguished as stated below.

2 Mathematical Thinking

Besides general educational goals, mathematics lessons are intended to help pupils develop specialized thinking abilities. As with all subjects at school, mathematics lessons aim to contribute to specific, important aspects in the development of thinking. But what is "mathematical thinking"? If you consider the brain as an organ of thinking, you can define it biologically as the following: *Mathematical thinking describes neurobiological processes, which are connected to dealing with mathematical situations.* But in order to create mathematics lessons, this definition needs further explication. It helps to differentiate the term "mathematical thinking" using a more detailed point of view. Figure 1 shows different aspects of mathematical thinking and distributes these aspects between three dimensions of thinking.

2.1 Content-based Thinking

Mathematical thinking occurs within various mathematical contents. These contents can be classified as follows:

- *Numerical Thinking:* e.g. developing and using the ideas of numbers, working with numbers, recognizing different aspects of numbers (cardinals, ordinals, measurements, etc.);
- *Geometrical Thinking:* e.g. developing terms and mental ideas regarding figures in plane and space, working with these figures, changing between two-dimensional to three-dimensional views, spatial perception, recognizing symmetries;
- *Algebraical Thinking:* e.g. learning and using calculating rules (Commutative Law, Associative Law, etc.), developing and using the idea of variables, working with mathematical terms and equations;
- *Stochastical Thinking:* e.g. understanding combinatorial situations, qualitative and quantitative probability, interpreting statistics; and
- *Functional Thinking:* e.g. understanding the relationships between causes and effects, working with functional relationships, e.g. with direct proportionality.

There are connections between the various content-based aspects of mathematical thinking. For example, numerical thinking is connected to geometrical thinking when someone thinks of numbers as points on the number line, as lengths or as vectors, or when someone visualizes calculations of numbers using geometrical images. Stochastical thinking is linked to numerical thinking if someone considers the counting principle of combinatorics as an application of multiplication or expresses probabilities using ratios.



Figure 1: Aspects of Mathematical Thinking

2.2 Process-based Thinking

Working on mathematics also refers to the process of thinking with mathematical structures. This mode of dealing with mathematics can be considered a second dimension of mathematical thinking that can be categorized as follows:

- *Experimental Thinking:* e.g. discovering mathematical phenomena, considering examples, changing situations, structuring and systematizing observations, explaining exploration-based ideas;
- *Concept-Formation Thinking*: e.g. generating general mathematical terms and concepts by analyzing examples (e.g. terms for categories of objects like "cone", terms for characteristics of objects like "centrically symmetric" or terms for relations between entities like "is a divisor of"), defining terms, classifying objects, forming generic terms and subsumable concepts, connecting terms;
- *Modeling Thinking*: e.g. analyzing situations by using mathematics, describing and working on situations through mathematical models, interpreting and evaluating the results;
- *Problem Solving*: e.g. working on problem situations, using problem solving strategies (such as systematical experimentation, restructuring of the situation and working backward), describing the solution in a clearly structured fashion;
- *Deductive Reasoning*: e.g. discussing and explaining ideas logically, creating or following chains of causality, understanding and developing proofs and proof techniques;
- *Formal Thinking*: e.g. operating with formal symbols such as digits or variables; thinking using mathematical symbols; dealing with formulas, terms and equations; and
- *Algorithmic Thinking*: e.g. using algorithms such as the standard addition and subtraction algorithms, applying specified methods of geometric construction – for example, constructing figures such that the end result has reflective symmetry.

These processes of dealing with mathematics cannot be strictly separated; they can occur in parallel and are often closely interlinked. For example, modeling processes can be based on a good concept formation,

or algorithmic or deductive reasoning can be part of problem solving actions.

2.3 Mathematically-based Information Processing

Mathematical thinking includes perceiving, managing, retaining and remembering mathematically-based information. These cognitive aspects of information processing constitute a third dimension of mathematical thinking, which can be further detailed as follows:

- *Mathematical Sensitivity:* e.g. perceiving mathematics in our environment, discovering special and interesting aspects of mathematical problems, detecting questions, grasping the structure of problems, familiarizing oneself with mathematical objects, recognizing aesthetic components of mathematics;
- *Thinking Using Mathematical Patterns*: e.g. finding general underlying patterns and structures within examples, abstracting, generalizing concrete situations, finding and using parallels between different situations, transferring general aspects to concrete situations, working with patterns;
- *Coping with Complexity:* e.g. extracting important information from complex situations, structuring information, easing processes by working in general terms, working with several objects at the same time;
- *Flexible Thinking:* e.g. switching between different levels of representation (enactive, iconic, verbal, mathematically symbolic), considering situations from different perspectives, restructuring situations, reversing conceptual processes;
- *Mathematical Creativity*: e.g. finding ideas and associations to mathematical situations, thinking divergently, going beyond existing limitations, using familiar information inventively, developing creative thinking, finding cross-connections;
- *Use of Language:* e.g. understanding situations which are presented verbally or written, using language to describe and present mathematical ideas and results; and
- *Mathematical Memory*: e.g. remembering mathematical situations and results, structures of reasoning and approaches to problems, connecting new information with existing knowledge, recalling knowledge for appropriate situations in a flexible way.

As with content- and process-based thinking, the different aspects of mathematically-based information processing cannot be considered separately. Thinking in mathematic patterns, for example, helps to cope with complexity and to remember certain content more easily. Creative thinking can be inspired by language and can be discussed by using language.

3 Mathematical Giftedness, Mathematical Ability, and Mathematical Performance

Through the different dimensions and aspects of mathematical thinking, it is easy to grasp the meanings of the terms "mathematical giftedness", "mathematical ability" and "mathematical performance".

3.1 Mathematical Giftedness

Mathematical giftedness is the individual potential to think mathematically. Let us have a closer look at different aspects of this definition. Mathematical giftedness is a potential. It is possible that persons have a certain potential to think mathematically, but they have not yet developed this potential. This may be because they have not yet been encouraged to do so by their environment. Primary school pupils, for example, can have a certain potential of think stochastically. But if they do not work on stochastic tasks, at school or at home, their potential in stochastics will not be nurtured and developed.

Mathematical giftedness is individually different. Persons differ according to their potential of mathematical thinking, as illustrated in Figure 1. High mathematical giftedness is characterized by an above-average potential in several aspects of mathematical thinking. This differentiated and multidimensional view of mathematical giftedness shows that it is impossible to describe this complex phenomenon by a one-dimensional scale. A simple reduction of mathematical giftedness to a number (such as IQ) does not reflect the complexity of mathematics. In order to assess pupils' needs at school and to give educational support, this multidimensional view of mathematical thinking is more appropriate.

3.2 Mathematical Ability

Mathematical abilities are the skills one has in order to think mathematically. The differentiated model of mathematical thinking in Figure 1 includes content-based thinking, process-based thinking and information processing. One may wonder how to fit mathematical knowledge and abilities to act mathematically into this model. Both fall under the conceptualization of mathematical ability. Mathematical knowledge is part of the concept of mathematical thinking, seen in all three dimensions in Figure 1. Accordingly, mathematical actions and mathematical thinking are closely related, because action should be led by thinking.

Mathematical abilities develop by dealing with mathematics based on mathematical giftedness. This process is influenced by a variety of individual characteristics and environmental features. The model depicted in Figure 2, illustrating the relationships between mathematical giftedness, ability and performance, includes all these aspects. This model is based on the "Münchner Hochbegabungsmodell" (meaning the Munich model of intellectual giftedness) by Heller (for example, see Heller 2001, Heller 2002, Heller & Ziegler 2007).



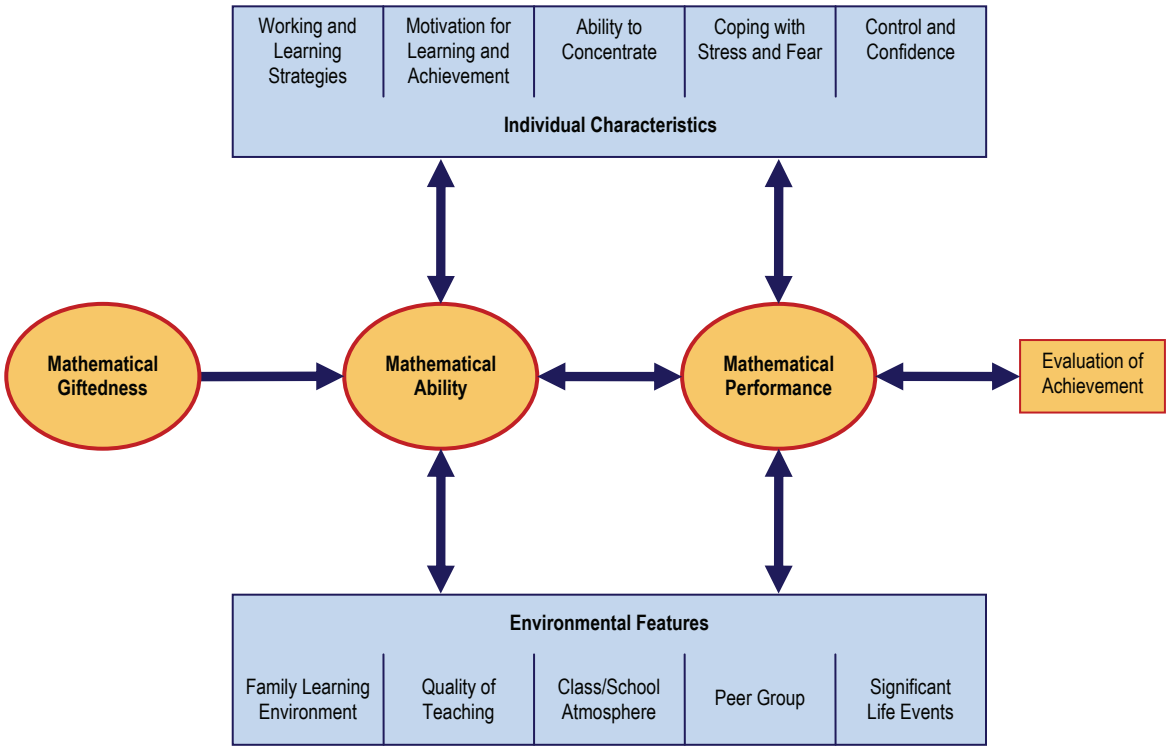


Figure 2: Giftedness, Ability and Performance

If and to which extent mathematical giftedness leads to mathematical ability is, in one way, determined by the individual characteristics of the pupil. These characteristics include:

- Strategies for working individually and independently, organizational skills,
- Learning and achievement motivation – specifically the willingness to make an effort, studiousness, stamina, intellectual curiosity, personal interest, personal outlook for the future,
- Concentration – meaning the ability to focus and concentrate on one situation so as to think and work carefully and in detail,
- The ability to cope with stress – particularly in managing nerves prior to and during an exam – and a positive attitude towards mistakes, and
- Confidence – meaning how far does one's own work lead to certain results and how much is one's effort worth, pertaining to feelings of self-efficacy and a person's self-concept.

On the other hand, one's environment plays an important role in ensuring that mathematical giftedness results in mathematical ability. These environmental factors are:

- Encouragement coming from the learning environment at home – including the type of education at home, expectations of achievement from home, the family's reactions to success and failure,
- Teaching quality – regarding different educational goals and the teacher's professional, didactic, pedagogic and social abilities,
- Atmosphere in class and at school – for example, concerning learning, achievement, social skills, and values,
- Influence of the peer group – including during free time activities,
- Significant life events – such as setbacks or successes.

Such individual characteristics and environmental features determine the way a person deals with mathematics, especially at school. They directly influence learning, mathematical abilities and performance.

3.3 Mathematical Performance

Mathematical performance describes the explicitly expressed results of mathematical thinking. More specifically, these results are exhibited in the pupils' expressed thoughts, notes in exercise books and on worksheets and written tests or exams. Such results require individual abilities and giftedness on the one hand, but on the other hand, they are also influenced by the complex system of individual characteristics and environmental features previously described and illustrated in Figure 2.

Only within social systems (such as school) can performance be appreciated, because only there are results of mathematical thinking discussed and eva-

luated by the social system. One must keep in mind that every evaluation of performance is based on criteria of the social system and cannot be totally objective. At school, mostly the teacher evaluates the student's performance by his or her pedagogic experiences, personal knowledge and particular individual values. This evaluation of performance provides feedback for the pupil, and therefore it influences pupil's future learning and performance.

4 Using This Model in the School

The model that is presented here can be useful for supporting and assessing pupils' needs and mathematical achievement at school, because it offers a detailed perspective on phenomena like "thinking", "giftedness", "ability" and "performance" in mathematics as a subject. It clarifies that mathematical giftedness does not lead automatically to mathematical abilities, and that mathematical abilities do not result directly in excellent performance. Conversely, a good performance does not prove high intellectual giftedness. The various influences stemming from personal and environmental features have to be considered.

This model also aids in assessing pupils' performance and achievement. Pupils' performance can be understood according to each of the different aspects of mathematical thinking described in Figure 1 and can be interpreted with respect to influential factors as illustrated in Figure 2. Furthermore, this model emphasizes that ability or giftedness can never be observed or measured directly, since every assessment can only attempt to draw conclusions from visible performance.

The detailed view of mathematical thinking offers a multidimensional perspective of pupils' abilities and giftedness. For example, a pupil might be good at algorithmic thinking but struggle with modeling; he or she might be strong in geometrical thinking but have difficulties in stochastical thinking. General statements such as "this student is poor in mathematics" seem to be unsubstantial and questionable.

Furthermore, this model can serve as a guide for supporting pupils – both individually and in class. Because the pupils' abilities of mathematical thinking are considered in a detailed manner, the pupils can be supported in a specific and individual way. For example, pupils who are poor at functional thinking need a specific learning environment in order for the pupils to fully develop their abilities. Pedagogy within the classroom should also include performance-enhancing characteristics and environmental features, so that a pupil's giftedness induces the best possible performance. Mathematics lessons should also focus on learning strategies, coping with fear and creating a positive class atmosphere.

Eventually, this model offers structure and help for planning general concepts for teaching mathematics – both in regular classes and in study groups for particularly gifted pupils. It helps educators to consider all different aspects of mathematical thinking and to teach regarding contents and methods, so that the pupils develop their abilities in varied and balanced ways. Furthermore, this book provides teaching materials needed to use this model in school.

Further Reading

Bardy, P. (2007): Mathematisch begabte Grundschulkinder, Diagnostik und Förderung, Spektrum Akademischer Verlag, Heidelberg

Bauersfeld, H., Kießwetter, K. (Hg., 2006): Wie fördert man mathematisch besonders befähigte Kinder?, Mildenberger, Offenburg

Gardner, H. (2001): Abschied vom IQ, Die Rahmen-Theorie der vielfachen Intelligenzen, Klett Cotta, Stuttgart

Heller, K. (Hg., 2000): Begabungsdiagnostik in der Schul- und Erziehungsberatung, Huber, Bern u. a.

Heller, K. (Hg., 2001): Hochbegabung im Kindes- und Jugendalter, Hogrefe, Göttingen

Heller, K. (Hg., 2002): Begabtenförderung im Gymnasium, Leske+Budrich, Opladen

Heller, K., Ziegler, A. (Hg., 2007): Begabt sein in Deutschland, LIT Verlag, Berlin

Käpnick, F. (1998): Mathematisch begabte Kinder, Lang, Frankfurt a. M.

Lack, C. (2009): Aufdecken mathematischer Begabung bei Kindern im 1. und 2. Schuljahr, Vieweg+Teubner, Wiesbaden

Peter-Koop, A., Sorger, P. (Hg., 2002): Mathematisch besonders begabte Grundschulkinder als schulische Herausforderung, Mildenberger, Offenburg

1 Exploring Magic Squares

Investigating Number Patterns Creatively

D. Bocka

Topic and Purpose

Magic squares are a special form of number patterns. The numbers are arranged such that all horizontal, vertical and diagonal sums are equal.

In primary school, this topic can be dealt with in all different grades. The aim is to discover the structures and patterns on which magic squares are based. Working with magic squares in class provides various exciting possibilities for investigating and discovering number patterns independently. While doing so, the pupils are not only supported in their creative and flexible thinking, but they also increase their addition and division skills. While working on the assignments, the pupils develop and use their own problem solving skills. The teacher then has the opportunity to discuss individual techniques with each child and the class as a whole. The following examples offer possibilities for interdisciplinary classes (arts, languages, etc.).

Background

Since ancient times, number symbolism has fascinated people. Oftentimes, single numbers, number sequences and number patterns have religious, astrological or political meanings. Several examples related to the topic of "magic squares" can also be found in literature and arts. The numbers within these magic squares are arranged such that the sums of each row, column, and diagonal are equal. This sum is called the "magic sum". To find the magic sum, add up all the numbers in the magic square, then divide the sum by the number of rows or columns. For example, if you have a 3x3 square with numbers from 1 to 9 you would:

- Add up the numbers: 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45,
- Divide the sum by the number of rows: 45 : 3 = 15,
- The magic sum is 15.

The smallest possible magic square with different numbers is a 3x3 square. If the numbers from 1 to 9 are used, there will be only one solution, which can be varied by symmetrical movements. Given below are examples of this, which are also solutions from the first worksheet.

2	9	4
7	5	3
6	1	8

2	7	6
9	5	1
4	3	8

4	9	2
3	5	7
8	1	6

4	3	8
9	5	1
2	7	6

6	1	8
7	5	3
2	9	4

6	7	2
1	5	9
8	3	4

8	1	6
3	5	7
4	9	2

8	3	4
1	5	9
6	7	2

To explain why there are no other solutions for a 3x3 square with numbers from 1 to 9, consider all the different ways you can add up three different numbers to equal 15. Systematically, the following eight sums are found: 1+5+9 = 1+6+8 = 2+4+9 = 2+5+8 = 2+6+7 = 3+4+8 = 3+5+7 = 4+5+6 = 15. These eight sums have to be found in each magical 3x3 square (three in the rows, three in the columns and two in the diagonals). These different sums reveal the structure of the magic square. First, observe that the number 5 occurs most often – four times in total – and therefore must be in the middle of the square. Consequently, the number 5 is part of one horizontal, one vertical and two diagonal sums. The even numbers occur three times and must be in the corners, so they are part of exactly one horizontal, one vertical, and one diagonal sum. The other odd numbers occur just two times, therefore they have to be in the middle of the outside rows and columns. They are only part of one horizontal and one vertical sum, but are not part of the diagonal sums. That is why the eight solutions shown above are the only possible answers for a 3x3 square that exist.

As a variation, you could examine 3x3 squares with other numbers. Are there magic squares which contain only even or odd numbers? The answer is yes, because you only have to double each number. Similarly, for the odd numbers, you only have to subtract 1 after producing the even-numbered squares.

Here are some general strategies used to create new magic squares from existing magic squares:

- Add a number to or subtract a number from each entry in the magic square.
- Multiply or divide all entries by a number.
- Add up all the entries of two magic squares, one by one.

By doing so, you can discover the change of the magic number. This topic will be a continuation of these worksheets.

Methodological Advice

The structure of the worksheets is directed towards reflection and independent discovery. Therefore, there should be enough time given for working on this topic. Collaborative work is recommended.
Every worksheet can be used independently after the initial explanation. A particular order is not necessary, and thus the topic "magic squares" can be covered at any time. The following list suggests one schedule for using these worksheets:

- Magic Squares for Beginners: from 1st grade
- Creating Magic squares: from 2nd grade
- The *Witch's One-times-one* by Johann Wolfgang von Goethe: from 3rd grade
- *Melancolia* by Albrecht Dürer: from 4th grade
- *Centennium* by Eugen Jost: from 4th grade

The magic square of the *Witch's One-times-one* is not a complete magic square. The numbers of each row and of each column sum to 15, but only one diagonal adds up to 15.
Single sections of this topic can be selected for deeper exploration. Some assignments refer to historical issues such as those regarding literature (Goethe's *The Witch's One-times-one*) and the arts (Dürer's painting *Melancolia*). The only part of Dürer's artwork that is printed in this book is the magic square, but it is worthwhile to look at the whole picture during the lesson. Examples of magic squares in contemporary art can be found in Baptist (2008) and on the internet: http://mathematik-und-kunst.de

For young pupils in particular, or for introducing the topic, the use of number cards with numbers 1 to 9 is recommended. Thus, the repeated use of numbers can be avoided, and wrong results do not have to be erased. However, the (partial) results should be written down. If you do not use square number cards, you should at least provide a blank 3x3 square consisting of small squares. These smaller squares can be filled in with the number cards. This allows symmetrical orderings to be quickly noticed.

It is quite difficult to notice that there is only one solution for a 3x3 square except for symmetrical orderings. Various orderings of the numbers 1 to 9 can be converted by rotation or reflection. For visualization, transparencies with blank 3x3 squares can be used, on which the different results can be written with a pen. By using an overhead projector, symmetrical properties can be easily examined. These properties become even clearer if the results are written as a dot pattern instead of numbers. These dots remain easily legible even when the square is turned and rotated.

Further Reading

Baptist, P. (2008): Alles ist Zahl, Kölner Universitäts-Verlag, Köln

Dahl, K., Nordqvist, S. (1996): Zahlen, Spiralen und magische Quadrate, Oetinger, Hamburg

Name: Date:

Magic Squares for Beginners

Magic squares had a special meaning in many advanced, ancient cultures. In ancient China, a certain magic square called Lo Shu (which means “document of numbers from the River Lo”) played a very important role. An old legend says that the Lo Sho was brought to the emperor by a turtle. On the turtle’s shell, the numbers one through nine were arranged in a three by three grid pattern such that the sums of the numbers in each row, column and diagonal were the same: 15. Such squares are called “magic squares”.

4	9	2
3	5	7
8	1	6

1. Fill in the following magic squares such that the result from adding up the numbers in each
   - row,
   - column, and
   - diagonal

   is 15. Use the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 only once.

2		
	5	3
	1	

2	7	6
4		8

4		2
	5	
8		

4		
9		1
	7	

6		8
	5	

	7	2
		4

3		
4	9	

		4
1		9

2. Examine the magic squares in more detail. What do you notice?

Name: Date:

Magic Squares for Beginners

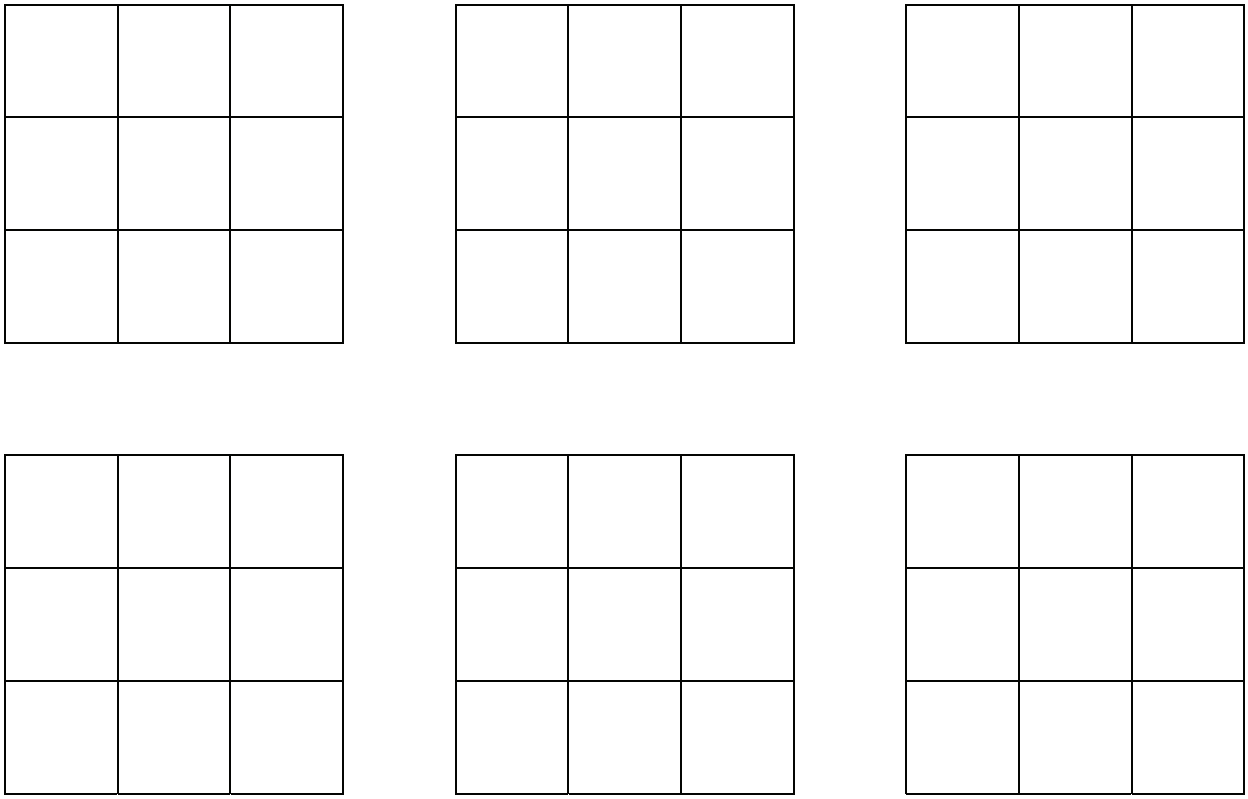
3. Which number is always in the same place? Why?

4. Take a closer look at the magic squares, and write down the numbers that you find in the corners. What do you notice? What do you call such numbers?

5. Take a look at the remaining numbers (in the middle of each side). Write them down. What do you notice? What are these numbers called?

6. Now write down the order of the numbers found in the outer little squares. Start in the top left-hand corner and continue in clockwise direction. What do you notice?

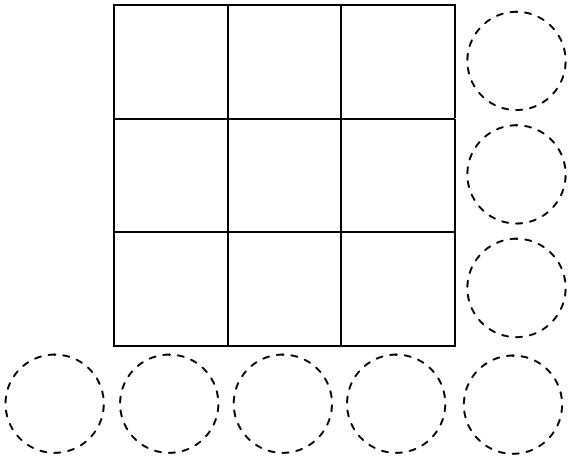
7. How many different solutions for 3x3 magic squares with the numbers 1 to 9 are there?



Name: Date:

Creating Magic Squares

1. Create a number square: Write the numbers 1 to 9, one after another, in the square. Start in the top left-hand corner with the number 1.
   Copy the pattern you created using number cards.

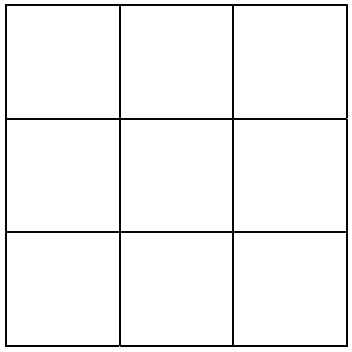


2. Add up the numbers in every row, every column and in both diagonals. Write down the results in the circles. What do you notice?

3. Add up the results of the three rows. Add up the results of the three columns. Add up the numbers from 1 to 9 as well. What do you notice? Explain.

4. Take a closer look at the numbers in the square:
   - Which numbers are written in the four corners? What do you call such numbers?
   - Which numbers can you find in the middle of the sides? How do you call these numbers?
   - Which number is in the center?

5. Change the number square: Erase all odd numbers in the corners. Leave the 5 in the center. Move all even numbers clockwise to the corners (2 to the right, 6 down, 8 to the left, 4 up). Describe the pattern.
   Add up the numbers in both diagonals. What do you notice?

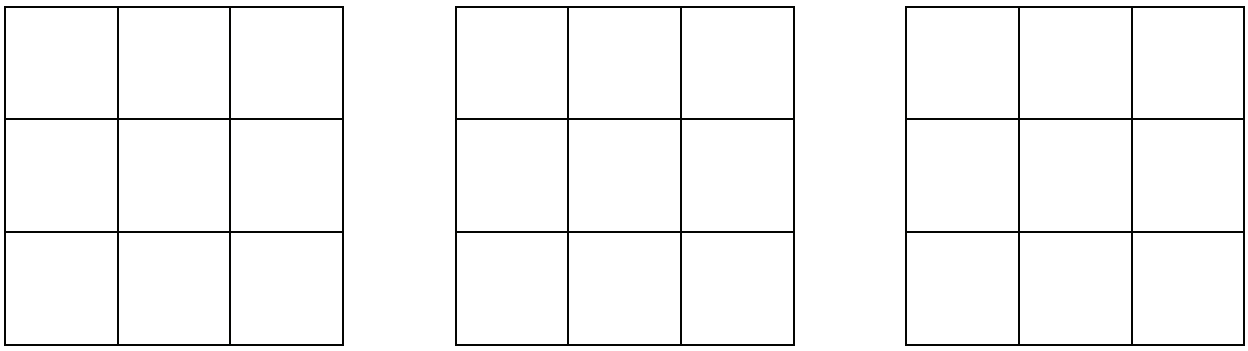


Name: Date:

Creating Magic Squares

6. Now write down the odd numbers 1, 3, 7, 9 in the square above so that each row and each column add up to the same result as the diagonals.

7. Move all even numbers in the square clockwise by two, four and six places. Where do you insert the odd numbers now?



What do you notice?

These number squares are called “magic squares”. The numbers within such a square are arranged such that the sums in each

- row,
- column, and
- diagonal

are equal.
This result is called the “magic sum”.

Name: Date:

The “Witch’s One-times-one” by Johann Wolfgang von Goethe

In Goethe’s play *Faust*, there are directions for creating a magic square. This passage is known as the *Witch’s One-times-one*, because a witch presents instructions for operating with the numbers. First read the whole text on the left. Then read the explanations on the right, and complete the magic square.

The Witch’s One-times-one	**Explanations**
Remember then Of one make ten,	Replace 1 by 10 in the first place.
The two let be,	Put 2 in the second place.
Make even three,	Put 3 in the third place.
Then rich you’ll be	Add up the numbers in the first row: Now you know that the magic sum is 15.
The four pass o'er!	Put 0 in the fourth place.
Of five and six, The witch so speaks, Make seven and eight,	Switch the 5 with the 7 and the 6 with the 8.
The thing is straight:	Put the missing number in the ninth place. The magic sum will help you.
And nine is one	The magic 3x3 square is a unit, because all the rows and columns have the same sum.
And ten is none –	A magic square with ten cells does not exist.
This is the Witch's One-times-one!	That is the name of the magic square.

Name: Date:

The “Witch’s One-times-one” by Johann Wolfgang von Goethe

1. You can create your own magic square of the Witch’s One-times-one.
   Start with the right square where the numbers 1 to 9 in the right order. Then change the order of the numbers according to the Witch’s One-times-one and the explanations.

1	2	3
4	5	6
7	8	9

2. Take a closer look at the magic square from the Witch’s One-times-one. Look at the numbers mentioned in the Witch’s One-times-one. What numbers are missing? Do you know why?

3. Compare the magic square of the Witch’s One-times-one with the following:

8	1	6
3	5	7
4	9	2

   What do you notice?
   Hint: Take a closer look at the diagonals.

Name: Date:

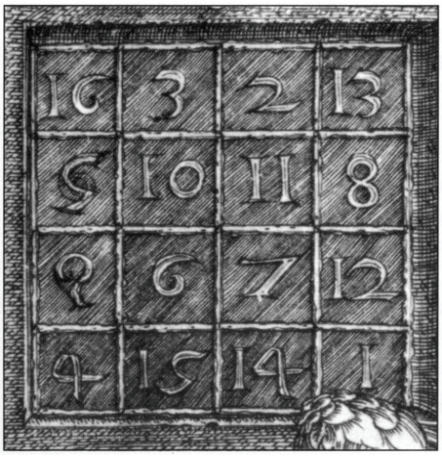
“Melancolia” by Albrecht Dürer

Names of Magic Squares

Magic squares consist of rows and columns. The smallest magic square that contains different numbers has three rows and three columns. You can call it either a “3x3 magic square” (pronounced: “three by three magic square”) or a magic square of order three. It consists of 3 times 3 = 9 small squares. A magic square with four rows and four columns is either called a 4x4 magic square or magic square of order four. It consists of 4 times 4 = 16 small squares.

The magic square by Albrecht Dürer

Look at the special magic square of order four given below. The artist Albrecht Dürer created a picture entitled *Melancolia*. In this picture you can see this magic square. (You can find the whole picture online or in books.) In order to make it easier for you to read the numbers in the magic square, the square is reproduced with a current font next to Dürer’s painting.



16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

1. Albrecht Dürer lived from 1471 to 1528. He hid the year the picture was created within the magic square. It is written in two small squares which are next to each other. In what year did Dürer create the painting?

   How old was he then?

   In the year of the picture, Dürer’s mother died at the age of 63. When was she born?

Name: Date:

“Melancolia” by Albrecht Dürer

2. Can you find the magic sum of Dürer’s square?

3. Dürer’s magic square is very special. Look at the small 2x2 squares in the corners and add up these numbers. What do you notice?

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

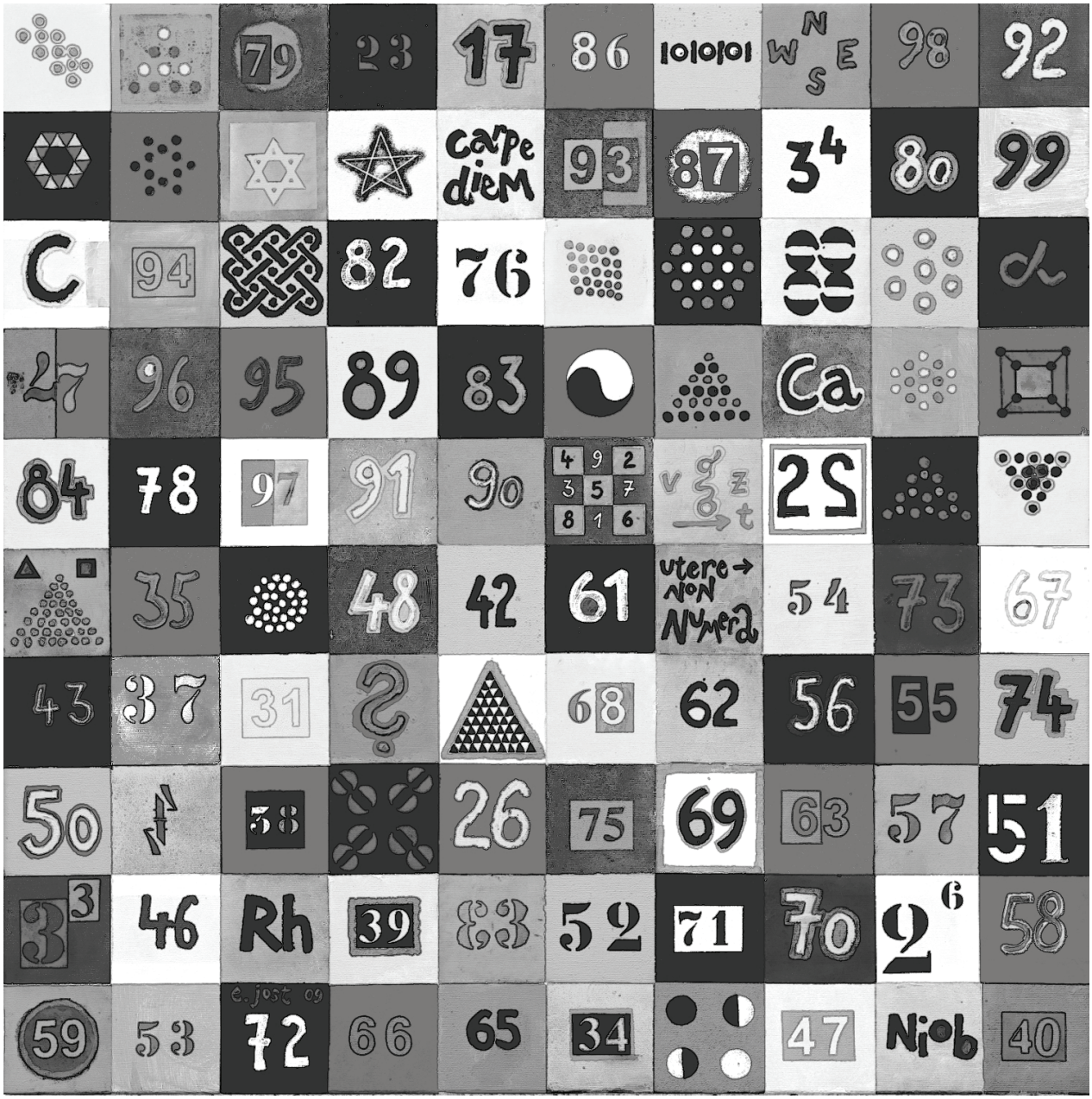
4. Do you find another 2x2 square with the magic sum within Dürer’s magic square? Color it in.

5. In Dürer’s magic square, the magic sum occurs even more often. Where else can you find it?

Name: Date:

“Centennium” by Eugen Jost

Look at the magic square by the Swiss painter and teacher, Eugen Jost.



1. Examine this picture and write down what you notice.

2. Jost used the numbers from 1 to 100 for his picture – but a lot of numbers are encrypted. Do you have an idea of what the magic sum might be?

Name: Date:

“Centennium” by Eugen Jost

3. The magic sum follows a certain rule: The sum of all used numbers is divided by the number of rows. Figure out why this rule applies to the magic sum. Then calculate the magic sum.

4. The picture is called *Centennium*. The name comes from the Latin word “centum” which means “one hundred”. Some of the numbers from 1 to 100 are encrypted. In the following grid you can see the unencrypted numbers from the picture. Complete the table. The magic sum will help you.

		79	23	17	86			98	92
					93	87		80	99
	94		82	76			88		
77	96	95	89	83					
84	78	97	91	90			22		
	35		48	42	61		54	73	67
43	37	31		49	68	62	56	55	74
50		38		26	75	69	63	57	51
	46		39	33	52	71	70		58
59	53	72	66	65	34		47		40

Some examples for solving the encryption:

- Figurate numbers (1st row, 1st column): Count the dots in the pattern.
- Roman numerals (3rd row, 1st column): C is 100
- Chemical elements (9th row, 3rd column): Rh is the abbreviation of the chemical element rhodium. It is in the 45th place in the periodic table.
- Number puzzle (1st row, 8th column): N, E, S, W are the abbreviations of all four directions: North, East, South and West. Therefore, the number 4 is encrypted.

2 Windows on the Hundreds Chart

Calculating and Exploring Structures

P. Ihn-Huber

Topic and Purpose

The hundreds chart represents numbers in a clearly structured way. The children choose numbers from the hundreds chart using windows cut out of paper and calculate with the numbers in the window. Therefore, they not only increase their calculation skills, but they also discover the structure of the hundreds chart.

Background

If you put Window 1, the window with three fields in a straight line, on the hundreds chart,



you will find that the number in the middle field multiplied by three is the same as the sum of all three numbers. Thus, you will find all multiples of 3 between 6 and 297 except for $3 \cdot 10$ and $3 \cdot 91$.

In Window 2, which consists of three fields in an L-shape, you have to consider several different cases. Depending on how the window is put on the hundreds chart, you get:



three times the number in the corner + 9



three times the number in the corner - 9,



three times the number in the corner + 11

or



three times the number in the corner - 11.

Thus, using this window, you get sums between 14 and 289.

Adding up the numbers in Window 3 provides the following results:

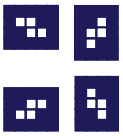


odd numbers from 45 to 359,



all even numbers from 18 to 386 which are not divisible by 4.

Summing the numbers in Window 4



provides only even results between 28 and 376.

In order to find the smallest or greatest result for each window, the children have to understand the structure of the hundreds chart. They must develop a strategy and consider all possibilities of turning and flipping the windows.

Moving the windows systematically provides results that change according to certain rules.
The sums of the numbers within the windows with three fields

- will increase by 3, if you move the window to the right (because each of the three numbers increases by one),
- will decrease by 3, if you move the window to the left (because each of the three numbers decreases by one).

The sum of the windows with three fields

- will increase by 30, if you move the window down (because each of the three numbers increases by 10),
- will decrease by 30, if you move the window up (because each of the three numbers decreases by 10).

The results of the windows with four fields follow the same rules, increasing or decreasing by 4 and 40, respectively.

In order to find three numbers that add up to a given result, children have to work strategically and purposefully by using what they already know about how the results change when they move the window in certain directions.

Methodological Advice

"Windows on the Hundreds Chart" can be a topic in class for two to three lessons.

Together the class as a whole can place Window 1 on the hundreds chart. The sum of the three numbers is calculated. Afterwards, the children should suggest what happens if you put the window in another place. ("The result increases." etc.) The teacher should instruct the pupils that they are allowed to turn and rotate the window. Then, the pupils can work individually on the first exercises on the first worksheet, which involve calculating and finding results that are as small or great as possible.

After a while, the teacher can guide the pupils with questions such as: What calculations did you do? What smallest or greatest result did you find? The teacher then collects the pupils' ideas and solutions.

When the pupils are familiar with this topic, they can work on the "Explorer's Tasks". They try out different windows and can choose the exercises themselves. The assignments can be posted on a wall where the children can hang up their solutions. The second discussion can be introduced by questions such as: Which exercises did you work on? What did you notice? Involving all the children's results there can be hold a "calculating conference", where various solutions are discussed. In the end, the children should reflect on other pupils' results by being posed the following questions: Which other children's discoveries amazed you? What else do you want to verify or explore by yourself?

Further Reading

Betz, B. u. a. (2010): Zahlenzauber 3, Bayern, Oldenbourg, München

Name: Date:

Hundreds Chart

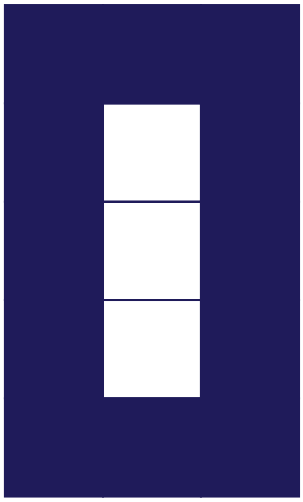
You can experiment with the hundreds chart.

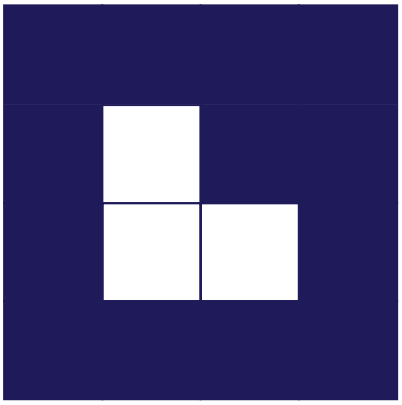
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

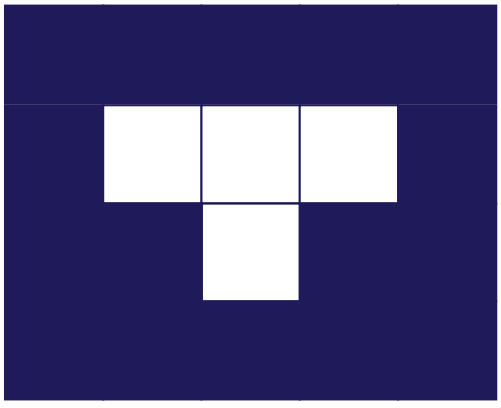
Name: Date:

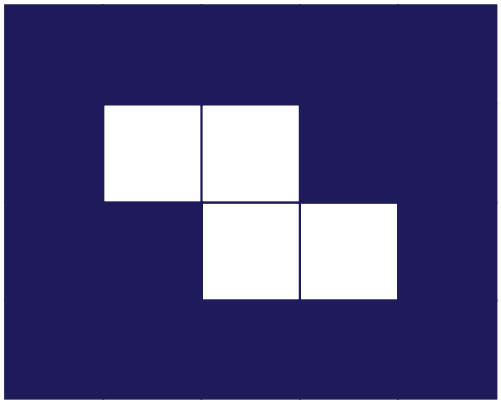
Windows for the Hundreds Chart

Cut out the four rectangles. Cut out the interior too. You get “windows” for the hundreds chart.



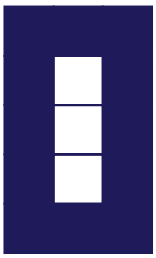






Name: Date:

Explorer's Tasks with Window 1



1. a) Put the window on the hundreds chart.
   Add up the numbers you see in the window.

   b) You can turn and rotate the window.
   Can you find the smallest and greatest possible results?

2. a) Put the window on the hundreds chart.
   Add up the numbers you see in the window.
   Move the window to the right or to the left by one field.
   What is the sum now?
   Compare your results. Can you figure out a rule?

If I move the window to the right If I move the window to the left

   b) What will happen if you move the window one up or down?

If I move the window up If I move the window down

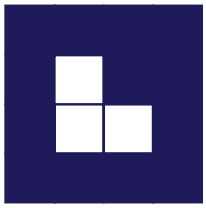
3. a) Try to put the window on the hundreds chart such that all three numbers add up to 45 (other sums: 144, 252, 102, 99, 33). Write down the calculations.

   b) Think of other results and try to find the three numbers in the window which add up to these results.

   c) For which results can you find these three numbers in the window; for which can you not find them?

Name: Date:

Explorer's Tasks with Window 2



1. a) Put the window on the hundreds chart.
   Add up the numbers you see in the window.

   b) You can turn and rotate the window.
   Can you find the smallest and greatest possible results?

2. a) Put the window on the hundreds chart.
   Add up the numbers you see in the window.
   Move the window to the right or to the left by one field.
   What is the sum now?
   Compare your results. Can you figure out a rule?

If I move the window to the right If I move the window to the left

   b) What will happen if you move the window one up or down?

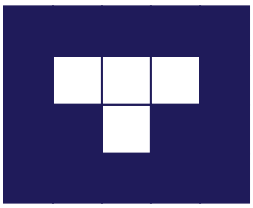
If I move the window up If I move the window down

3. a) Think of other results and try to find the three numbers in the window which add up to these results.

   b) Are there any sums that you cannot add up to?

Name: Date:

Explorer's Tasks with Window 3



1. a) Put the window on the hundreds chart.
   Add up the numbers you see in the window.

   b) You can turn and rotate the window.
   Can you find the smallest and greatest possible results?

2. a) Put the window on the hundreds chart.
   Add up the numbers you see in the window.
   Move the window to the right or to the left by one field.
   What is the sum now?
   Compare your results. Can you figure out a rule?

If I move the window to the right If I move the window to the left

   b) What will happen if you move the window one up or down?

If I move the window up If I move the window down

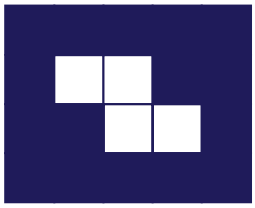
3. a) Try to put the window on the hundreds chart such that all four numbers add up to 70 (other sums: 142, 374, 233, 99, 42).

   b) Think of other results and try to find the four numbers in the window which add up to these results.

   c) Do you find solutions for all numbers between 105 and 110?

Name: Date:

Explorer's Tasks with Window 4



1. a) Put the window on the hundreds chart.
   Add up the numbers you see in the window.

   b) You can turn and rotate the window.
   Can you find the smallest and greatest possible results?

2. a) Put the window on the hundreds chart.
   Add up the numbers you see in the window.
   Move the window to the right or to the left by one field.
   What is the sum now?
   Compare your results. Can you figure out a rule?

If I move the window to the right If I move the window to the left

   b) What will happen if you move the window one up or down?

If I move the window up If I move the window down

3. a) Think of other results and try to find the four numbers in the window which add up to these results.

   c) For which results can you find these four numbers in the window, for which sums can you not find four numbers in the window?

3 Exploring the System

Experimenting with Number Sequences

M. Brandl, B. Brandl

Topic and Purpose

This set of tasks focuses on number sequences. The pupils analyze the structure of sequences, find rules, describe them and continue the sequences. In some examples you can switch between sequences of numbers and sequences of geometric figures. Rules can be defined explicitly (e.g. triangular and square numbers) or recursively using the previous numbers of the sequence (e.g. Fibonacci numbers). The worksheets also offer tasks that involve deciding if a certain number belongs to a sequence or does not.

Background

The sequences of this chapter can be presented in an illustrative way, which is easy to remember.
On the one hand, there are "figurate numbers" such as triangular, square and rectangle numbers. They correspond to regular geometric shapes or geometric patterns of certain triangles, squares and rectangles. On the other hand, there is the Fibonacci sequence, which can be introduced in a historical context regarding the "rabbit problem" (see the chapter titled *The Postman and Fibonacci*).
Examples of explicit and recursive formulas are shown below.

	Explicit formula		Recursive formula
Triangular numbers	$D_n = \frac{1}{2} n(n+1)$	$D_n = 1 + 2 + 3 + \ldots + n$	$D_n = D_{n-1} + n$
Square numbers	$Q_n = n^2$	$Q_n = 1 + 3 + 5 + \ldots + (2n-1)$	$Q_n = Q_{n-1} + (2n-1)$
Rectangle numbers	$R_n = n(n+1)$	$R_n = 2 + 4 + 6 + \ldots + 2n$	$R_n = R_{n-1} + 2n$
Pyramid numbers	$P_n = \frac{1}{6} n(n+1)(n+2)$	$P_n = D_1 + D_2 + D_3 + \ldots + D_n$	$P_n = P_{n-1} + D_n$
Fibonacci numbers			$F_n = F_{n-1} + F_{n-2}$

Furthermore, there are interconnections between different sequences such as:
$2D_n = R_n$ or $D_{n-1} + D_n = Q_n$
The pupils can explore, describe and even give reasons for most of those connections by geometric pictures – of course without using any variables and formulas.

Methodological Advice

The children could create the geometric patterns with chips or dice. An alternative is working with matches (40 matches per pack). This also provides the opportunity to approximate the number of matches in one box, and thus to extend the topic to include statistics (Spiegel 1982).

Further Reading

Müller, G., Steinbring, H., Wittmann, E. (Hg., 2004): Arithmetik als Prozess, Kallmeyer, Seelze

Spiegel, H. (1982): Wie viele Streichhölzer sind in einer Streichholzschachtel? Mathematische Unterrichtspraxis 3, Heft 1, p. 3-14

Name: Date:

Sequences of Numbers

1. Leo wrote down the following numbers. Think of a rule for each sequence and continue each sequence systematically.

   a) 240 232 224 ___ ___ ___ ___ ___ ___

   b) 5 25 6 36 ___ ___ ___ ___ ___

   c) 2 4 8 ___ ___ ___ ___ ___ ___

   d) 422 427 407 412 ___ ___ ___ ___ ___

2. Describe the rule for each sequence in your exercise book.

3. Kathryn claims that the numbers 256, 18, 100, 382, 81, 120 and 1028 will also occur in Leo's sequences if you write down enough numbers. Is she right? If she is right, to which sequence does each number belong? Give reasons for your choice.

4. Think of your own examples of sequences and describe the rules for them.

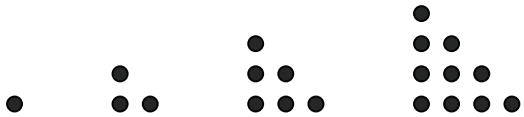
Name: Date:

Triangular Numbers

1. Look at the following numbers, figure out the rule and continue the sequence.

   1 3 6 10 ___ ___ ___ ___ ___ ___

2. Anna has created the following sequence of patterns with small chips.



   What is the connection between Anna's sequence of patterns and the numbers above in exercise 1? Write down your ideas in your exercise book.

3. Continue the sequence by creating the next patterns with small chips. Draw the patterns you found in your exercise book.

4. Explain why these numbers are called "triangular numbers".

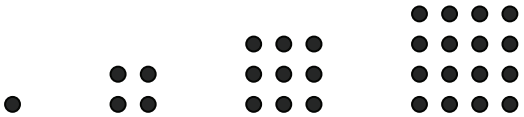
5. Do the numbers 32 and 36 occur in this sequence? Explain your answer.

6. Continue exploring the sequence and describe your observations.

Name: Date:

Square Numbers

1. Max has created the following sequence of patterns. Draw the next three patterns in your exercise book.



2. Which sequence of numbers matches the sequence of patterns? Continue the sequence for three more numbers. Do you have an idea what the name of these numbers might be?

3. Ken claims that the eighth pattern consists of 64 chips, because:

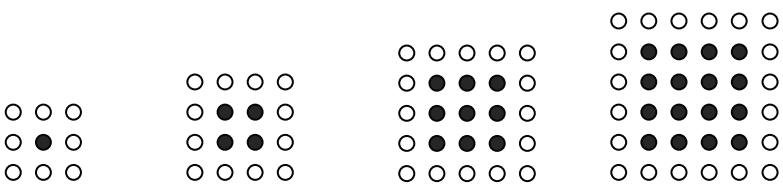
$$64 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15.$$

Explain how Ken figured this out.

4. Write the numbers 81 and 100 as a sum like Ken did.

5. Max tells Ken it is much easier to calculate the numbers of the chips. What does Max mean?

6. Ken expands the original sequence of patterns by creating a frame with white chips around the patterns. Write down the number of the white chips used as a sequence of numbers for the first ten patterns. Explain how you calculate these numbers.



Name: Date:

Rectangular Numbers

1. Find the rule for the sequence and continue the sequence systematically.

   2 6 12 20 ___ ___ ___ ___ ___ ___

2. These numbers are called “rectangular numbers”. Can you explain the name by drawing the right patterns using chips for this sequence?

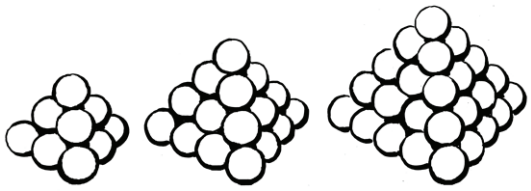
3. There is a way to use multiplication to describe each rectangular number. Explain.

4. Do the numbers 62 and 72 belong to this sequence? Explain your answer.

5. Think of similar tasks and share them with your neighbour.

Pyramid Numbers

1. Emily builds a pyramid out of balls. Draw the next smaller pyramid and the next bigger one:



2. How many balls do you need to build each of these pyramids? Write down the number of balls for the first seven pyramids and explain how you know.

3. Describe how you can calculate the number of balls for each pyramid.

4. Betty claims that there will not be any balls left, if she builds the sequence of pyramids out of 209 balls. What do you think?

5. Explore the connections between triangular numbers, square numbers, rectangle numbers and pyramid numbers.

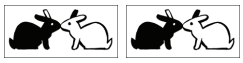
Name: Date:

Fibonacci Numbers

In 1202 B.C. the book *Liber Abaci* by Leonardo of Pisa, also known as Fibonacci (Bonacci's son), was published. There you can find a well-known problem involving the population growth of rabbits:

"A pair of rabbits produces one new pair of rabbits every month from the second month on. The descendents do the same. The rabbits never die."

In the table below, you find the number of pairs of rabbits for the first four months:

beginning	after the 1st month	after the 2nd month	after the 3rd month	after the 4th month
		 	 	  

1. Explain the numbers of rabbit pairs. Create the sequence with chips, and continue the sequence. Let each chip represents one pair. Write down the number of chips as a sequence of numbers.

2. Claudia looks at the sequence and says: "That's quite simple: For finding the next number, you just have to ..." What does she mean?

3. Do the numbers 49 and 55 belong to the sequence? Explain your answer.

4 The Postman and Fibonacci

Exploring the Fibonacci Sequence

D. Bocka

Topic and Purpose

The Fibonacci numbers are one of the most interesting and diverse number sequences. They are not only worthy of mathematical consideration, but they also appear in nature in the spiral arrangements of leaves and petals or in the design of certain plants. The number sequence starts with 1, 1, 2, 3, 5, 8, 13, 21, 34 and continues according to a specified rule: each subsequent number is the sum of the previous two.

The following problem, "The Postman", leads to the Fibonacci sequence. "A postman is walking up the stairs. He takes either one or two steps at the same time, and he always takes the first step. How many possibilities does he have to walk up to the first, second, third, fourth, fifth, sixth, ... step?" This problem helps, particularly in the case of young pupils, to investigate the number sequence in an exploratory way.

Background

Leonardo of Pisa (ca. 1180 – 1240), known as Fibonacci, posed the following puzzle in his mathematical treatise *Liber Abaci*: "A newly born pair of rabbits is put in a field. They are able to mate at the age of one month so that at the end of the second month a female can produce another pair of rabbits. The rabbits never die and a mating pair always produces one new pair of rabbits every month from the second month on. How many pairs will there be after one, two, three, four, ... months?" This is the oldest known source of the Fibonacci sequence. The number of the rabbit pairs 1, 1, 2, 3, 5, 8, 13, 21, 34, ... follows this rule: the two previous elements add up to the subsequent element. This can be understood in the following way. The number of the rabbit pairs in a certain month is equal to the number of the pairs from the previous month plus the number of the newly born rabbits. The latter is equal to the number of pairs that were already alive two months ago and therefore can be parents now.

The problem about the postman leads to the same sequence. How many possibilities are there for the postman to walk up to a certain step? For his last step he has two possibilities, he takes either one or two steps. Thus, you only have to add up the number of possibilities for him to walk up to the second to the last step and the number of possibilities for him to walk up to the third to the last step. The basic mathematical structure is the same as the one about the growth of the rabbit population.

Recently, the first eight Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21 became well-known by Dan Brown's novel *The Da Vinci Code* (2003) and by the movie (2006). In this story these numbers, at first presented out of order, are arranged into the right order and are the code for a safe deposit box.

Methodological Advice

If you would like to introduce the topic of Fibonacci numbers to a class, several groups can be built. One child can be the "postman" and up to six other children can be the "guards," each of who stands on one step. The children mark each postman's walk with different coloured cards on the steps. The "guards" put the cards ("footprints" made out of coloured paper) on the step when the postman steps on it. Either the "postman" or an additional "writer" notes down the number of possibilities on the worksheet. An alternative is that all children make notes on their worksheets.

Approaching the Fibonacci numbers via the growth of the rabbit population is described in this book under the section titled *Exploring the System*. It is recommended to work on that section just before or after working on the postman problem.

Further Reading

Baptist, P. (Hg., 2008): Alles ist Zahl, Kölner Universitäts-Verlag, Köln

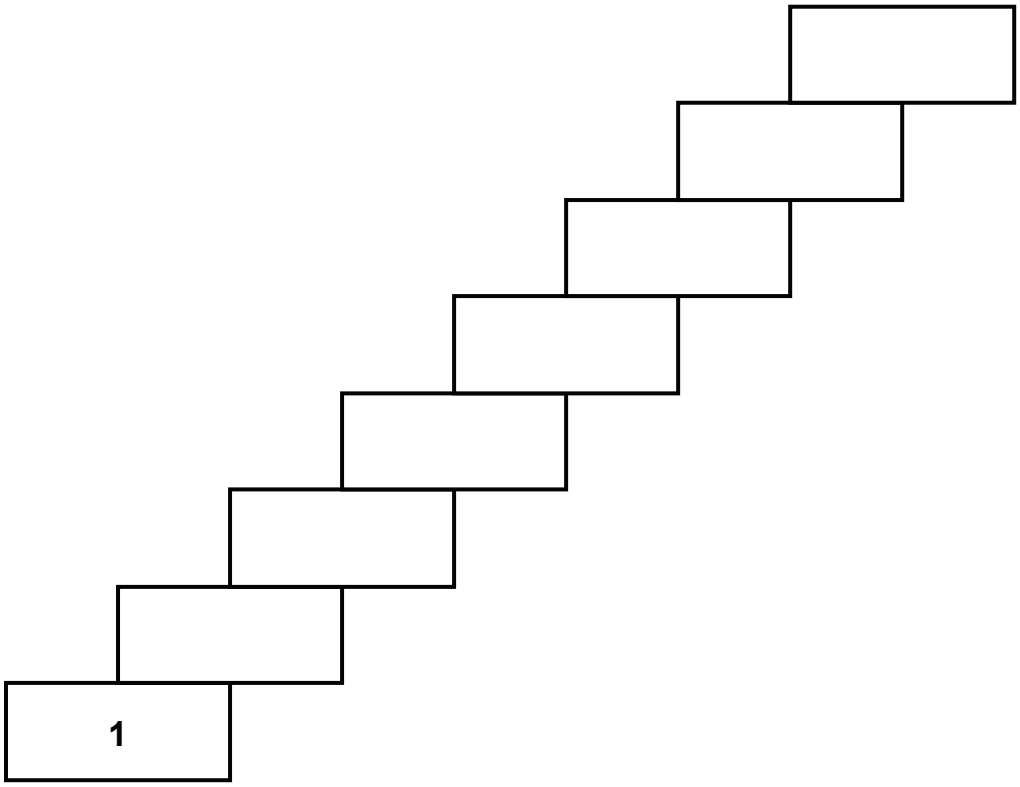
Schweingruber, T. (2006): Auf zum MATHerhorn – Spannende Mathematik für Kinder, Sauerländer, Zürich

Name: Date:

The Postman and Fibonacci

A postman is walking up the stairs. He takes either one or two steps at the same time, and he always takes the first step. How many possibilities does he have to walk up to the first, second, third, fourth, fifth, sixth, ... step?

1. Try out all the possibilities on the stairs.
   For every step in the following drawing, write down how many possibilities there are to walk up to that step.



2. What do you notice? Describe your observations in your exercise book.

3. Which numbers come next in the sequence? Describe how you calculated the next numbers. Why does your method for calculating the numbers work?

4. Collect information about the Italian mathematician Fibonacci. What does the postman problem have to do with him?

5 Experimenting with Number Cards
Combining and Calculating

K. Weigl

Topic and Purpose

While experimenting with number cards from 0 to 9, the pupils should learn about the concept of numbers, the structure of numbers and the decimal system simply by handling and working with numbers. Because the structure is important for all ranges of numbers, all grades in the primary school can work on this topic. The teacher or the pupil only has to decide which numbers (two-, three-, four-digit) he or she would like to work with.

The pupils should obtain an in-depth understanding about numbers and their structure, and they are expected to perceive and differentiate between calculations beyond the basic application of algorithms, which may occur without understanding.

Creating different numbers and calculations begs the question: how many possible combinations are there? While working on this topic, the pupils develop different systematic ways of solving the problems, which are very important for their individual learning process.

Background

Creating numbers by using number cards from 0 to 9 is a task consistent with the place value system. Thus, the pupils rediscover the rules that there can be no number greater than 9 in any column and that you must not omit the number 0. Furthermore, by working on certain value places, the pupils comprehend the particular significance of the individual positions in comparison to adjacent positions. To find a number which is between 500 and 1000, the pupils have to start looking at the hundreds position, whereas they could look at the thousands position in order to find a number between 500 and 5000. The children can work on the exercises with just one set of number cards, and thereby the focus is on the structure of the numbers.

The exercises that are connected to certain calculations and their solutions are not primarily aimed at practicing calculation skills but more at improving the pupils' understanding of connections within the place value system. For example, in order to create two three-digit numbers whose difference is as close as possible to 300 using the digits 0 to 5, the pupils could first choose the hundreds (5/2, 5/3, 4/1, 4/2, …). While doing so, the pupils have to decide if the result they get is more or less than the target number. Then they must approach the desired difference step by step by using the digits in different ways. While trying and calculating in a structured way, they have to switch from isolated place values to the whole system of numbers and vice versa. The pupils' suggestions and written results clearly show their degree of abstraction and the level of structured mathematical thinking.

The teacher only tells the pupils which number cards they have to use, but not the position and thus the exact values. Therefore, the child can choose the numbers that match his or her level of mathematical proficiency.

Methodological Advice

In order to work on this topic, each student needs a set of number cards from 0 to 9. He or she can create different numbers and calculations by him- or herself. However, the possibilities are limited, because each number card can only be used once. In order to prevent unsystematic guessing, it is important for the pupils to note every attempt in their exercise books. While creating both certain calculations and exercises using combinatorics, written notes provide specific points to compare and discuss in class. The pupils gain a significant amount of knowledge by looking back at the results and discussing their solutions. Thus, there should always be a discussion while working with this topic. You cannot control the solutions by just comparing them, but you can discuss them in class.

The task to invent similar exercises supports individual learning, because every child can choose exercises that support his or her abilities. It encourages the pupils to further work on this topic by using previous knowledge and strengthening creative thinking, especially when working in pairs with two sets of number cards.

This section on number cards offers three levels of difficulty. The different worksheets are recommended for the following grades:

Experimenting with Number Cards (1)	from 1st grade
Experimenting with Number Cards (2)	from 2nd grade
Experimenting with Number Cards (3)	from 3rd grade

Of course, it might be useful to use different levels within one class and match the tasks according to the individual pupils' mathematical proficiency.

Further Reading

Hirt, U., Wälti, B. (2008): Lernumgebungen im Mathematikunterricht, Kallmeyer, Seelze

Gasteiger, H. (2008): Addition bis 100 mit Ziffernkärtchen, in: Ulm, V.: Gute Aufgaben Mathematik, Cornelsen Scriptor, Berlin

Name: Date:

Number Cards

4	9
3	8
2	7
1	6
0	5

Name: Date:

Experimenting with Number Cards (1)

You each need the number cards **0 1 2 3 4 5 6 7 8 9**.
While experimenting, write down every number and every calculation you find in your exercise book.

1. Choose four number cards (except for 0).
   Note down all the two-digit numbers you can create with the number cards.
   How many can you find?

2. Find pairs amongst your numbers that are close to each other.
   What do you notice? Explain.

3. Find pairs amongst your numbers that are far from each other.
   What do you notice? Explain.

4. Now, choose four other number cards.
   Complete tasks 1 through 3 again, but before doing so, guess what will happen.

5. What is different when one digit is 0? Explain.

6. Which four cards do you have to choose so that the difference between the largest and the smallest two-digit number is as small as possible?

Name: Date:

Experimenting with Number Cards (2)

You each need the number cards **0 1 2 3 4 5 6 7 8 9**.
While experimenting, write down every number and every calculation you find in your exercise book.

1. Choose four number cards (except for 0).
   Write down all two-digit numbers you can create with the number cards.
   How do you know you do not forget a possible two-digit number?

2. Find pairs amongst your numbers for which the difference is as small or as large as possible.

3. Create numbers with the cards that are as close as possible to the middle of 10 and 70.

4. Find pairs amongst your numbers with the sum as close to 50 as possible.

5. Find pairs amongst your numbers for which the difference is as close to 20 as possible.

6. Create your own tasks like exercises 4 and 5 above, and work on them.

7. Find which three-digit numbers you can create with your cards. Examine these in detail.

8. What is different when one digit is 0? Explain.

9. Now choose four other number cards and repeat tasks 1 to 8.
   Before doing so, guess what will happen.

Name: Date:

Experimenting with Number Cards (3)

You each need the number cards **0 1 2 3 4 5 6 7 8 9**.
While experimenting, write down every number and every calculation you find in your exercise book.

1. Choose six number cards (except for 0).
   Write down three-digit numbers you can create with the number cards.
   How many numbers could you create? Explain.

2. Create pairs of three-digit numbers with your number cards for which the sum is as large or as small as possible.

3. Create pairs of three-digit numbers for which the difference is as large or as small as possible.

4. Create numbers that are as close as possible to the middle of 200 and 800.
   Look for numbers that are as close as possible to the middle of 220 and 620.

5. Create pairs of numbers with your cards for which the sum is as close as possible to first 500, then 1000.

6. Create pairs of numbers with your cards for which the difference is as close as possible to first 500, then 200.

7. Create your own tasks like exercises 5 and 6 above, and work on them.

8. How many four-digit, five-digit and six-digit numbers can you create with your six number cards?

9. Now, choose six other number cards and repeat exercises 1 to 8.
   When choosing the number cards, pay attention to creating numbers with as small or as large sums or differences as possible.

10. How many four-digit, five-digit and six-digit numbers can you create with all ten number cards?

6 A World Where People Have Only Six Fingers

Exploring Base Six as a Place Value System

R. Motzer

Topic and Purpose

What if we only had three fingers on each hand? How would we calculate? In this section, a fictional story shows the pupils such a world, where initially a lot of things seem strange. The children should discover by themselves the unusual use of the numbers, their origin and how to calculate with those numbers.
At first it is "illogical and confusing" for pupils (quoted from several fourth-grade pupils). But by thinking about this number system, they can discover the logic of the base-six system. Familiar tasks (such as calculating with number walls) should be worked on in the base-six system. With this system, the pupils are able to work out the idea of bundling in the construction of numbers. They have probably already become used to the decimal system, so that they no longer question it. Due to the foreign of base-six, they become aware of basic principles of arithmetic.

Background

The section is based on the place value system. The ability to write numbers as numerals, to use the bundling principle and to interpret combinations of these numerals is very important. The value of single digits and also the order in which the digits are written is of great importance, because the position of a digit determines its value.
To understand the base-six system, which is used in the fictional world, it is important to grasp the idea that the base-six system regroups into bundles of six: 6 ones (units) form a group of six, 6 groups of six form a group of thirty-six, and 6 groups of thirty-six form a group of two hundred and sixteen, and so on. It is recommended to use the subscript "6" to indicate the difference in the decimal representations.

The sequence of natural numbers in the base-six system is:
1_6, 2_6, 3_6, 4_6, 5_6, 10_6, 11_6, 12_6, 13_6, 14_6, 15_6, 20_6, 21_6, 22_6, 23_6, 24_6, 25_6, 30_6, …

The following examples show how to convert numbers from the base-six into the base-ten system according to the bundling principle mentioned above:

$34_6 = 3 \cdot 6 + 4 \cdot 1 = 22_{10}$

$215_6 = 2 \cdot 36 + 1 \cdot 6 + 5 \cdot 1 = 83_{10}$

$100_{10} = 2 \cdot 36 + 4 \cdot 6 + 4 \cdot 1 = 244_6$

Addition and subtraction can be done in each place value position, as with the decimal system:
$103_6 + 54_6$ = 1 thirty-six + 5 six + 7 one =
2 thirty-six + 0 six + 1 one = 201_6

Methodological Advice

This topic is best suited for highly gifted pupils. The pupils read the story and try to understand the situation. They can work either individually or in groups. Afterwards, they can solve the tasks of the school of the green-haired children in order to develop and consolidate understanding of the base-six system. Additionally, the children should work with teaching materials (they can even prepare these materials themselves) such as "one cube" (1cm x 1cm x 1cm), "six poles" (1cm x 6cm x 1cm), "thirty-six flat squares" (6cm x 6cm x 1cm), and, if there is enough material, "two hundred and sixteen cubes" (6cm x 6cm x 6cm).
Further possibilities are working with the number line and with the 36-square chart. Creating such teaching materials helps to increase awareness of corresponding materials in the base-ten system.

Name: Date:

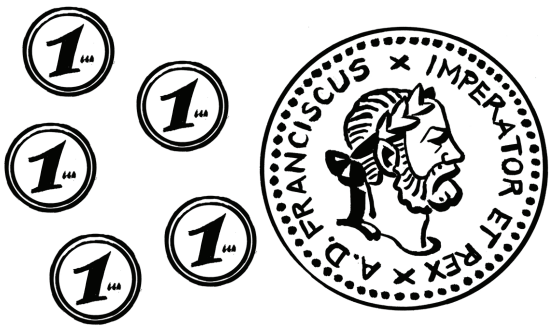
The Dream

It was Monday morning. Liz was on her way to school. She was still a bit tired, when she met her best friend, Ben. “Hey Liz! I have to tell you something!” he shouted from across the street. “Last night I had a very strange dream. You were in it too! We went to another country on a plane. Everything was totally different there!”

Liz was amazed and exclaimed, “Wow, I have never been on a plane before! What was it like in the foreign country? Did we like it?” “Yes, it was great, but quite different to life here. People had green hair for example!” Ben answered. “Come on! Tell me! What was different?” Liz asked impatiently. Ben started to talk about his dream and the experiences he had in the other country.

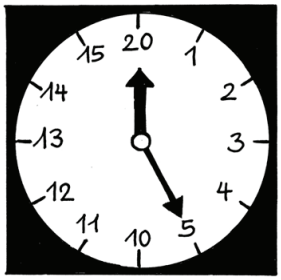
“Suddenly, we were in a town. At first, everything seemed to be normal. Then we came to a market place. There were strange people with green hair everywhere! On the right, there was a market stall where you could buy all kinds of vegetables. The stall keeper wrote every kilogram he sold on a small blackboard next to him. But somehow it looked really strange – somehow different from our way of writing numbers!”





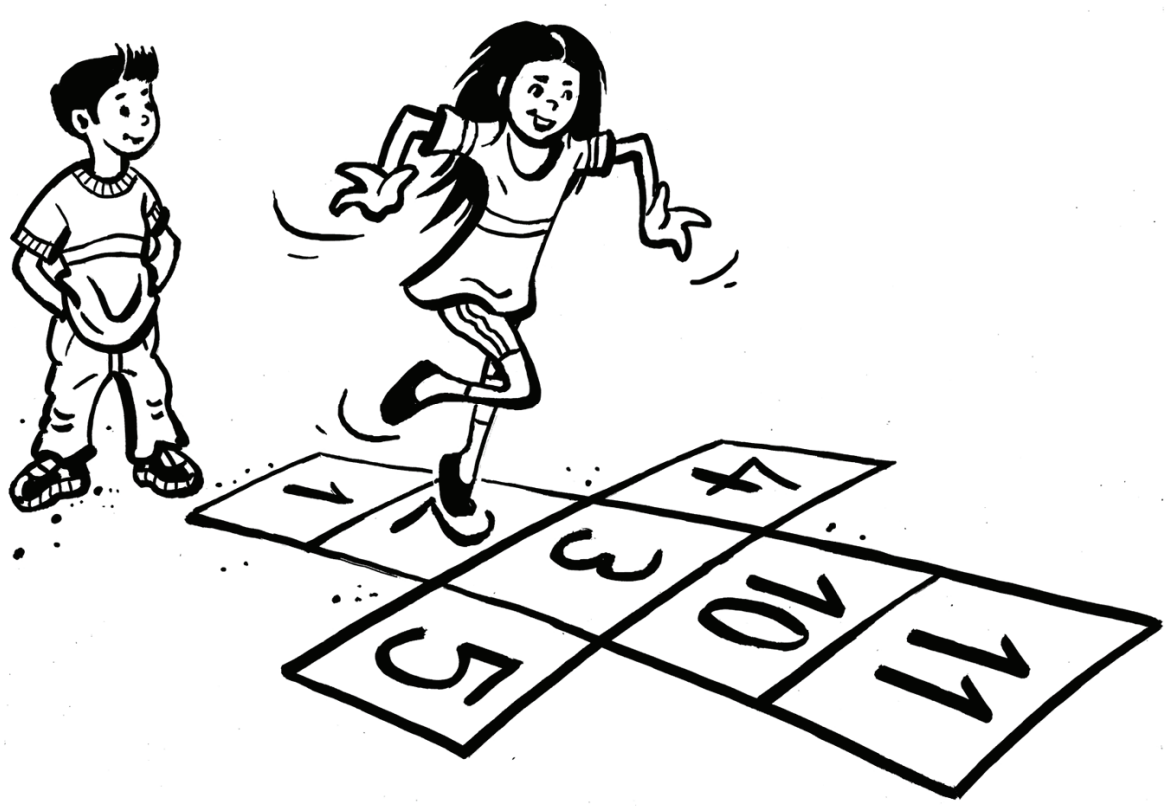
“Guess what! The money was strange, too. Small coins were called ‘coppers’ and the big ones ‘guilders’. Six coppers made one guilder.”

Ben continued, “On the big square in front of the church some children were playing hopscotch. They had to hop from one square to another, one by one. One of the children was counting loudly: One, two, three, four, five, one-zero, one-one, one-two...”
“They can’t count!” Liz interrupted him.



Ben went on, “There is more to it than that! Suddenly the big church clock began to chime. The clock looked so strange!”

Name: Date:



“For some reason we had to catch the next bus. When we arrived at the bus station I looked at my watch. It was 18:50. So then I went to the time table to see when the next bus would arrive. But it just said some random numbers. I couldn’t find 18:50.”

Section from the time table of bus 110

25:00	25:113
30:00	30:113
31:00	31:113
32:00	32:113

“What happened in the end?” Liz asked in amazement. Ben said, “We finally caught a bus and went to a birthday party. There we saw the strangest thing ever! In this country people only have three fingers on each hand!”

Name: Date:

“Gosh, what strange dreams you have! As if such a world really existed! Come on now. We have to hurry or we’ll be late for school!” Liz gazed at Ben shaking her head in amazement and ran off towards the school.



At school both of them thought about how children would calculate in this foreign place. Would they know, for example, number charts and what would they look like?
Would they need as much time as we need for learning the times-tables by heart?

Exercises for the Dream

1. What do you notice in this story?

2. How do people count or calculate in this strange place? What is the difference from our way of counting or calculating?

3. What does the number line look like in that country?

4. What are the departure times of the bus on our clock?

5. Would anything similar to the 100-square chart exist in that dream land?

(The story was written by Anja Reiter, Brigitte Schnitzer and Angela Scolaro.)

Name: Date:

From a School Book in Dream Land

1. Continue the number sequences and describe the system on which they are based.

1_6 / 3_6 / 5_6 / 11_6 / 13_6 /

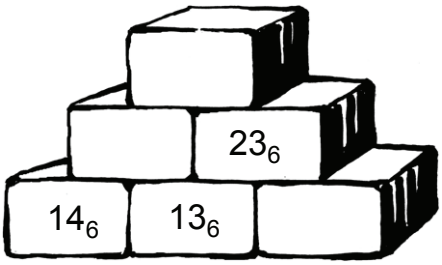
10_6 / 20_6 / 30_6 / 40_6 / 50_6 /

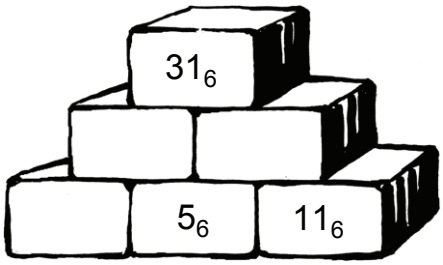
3_6 / 10_6 / 13_6 / 20_6 / 23_6 /

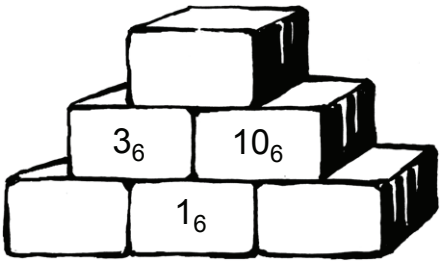
10_6 / 3_6 / 20_6 / 13_6 / 30_6 /

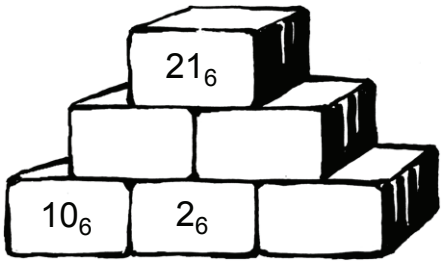
10_6 / 4_6 / 13_6 / 11_6 / 20_6 /

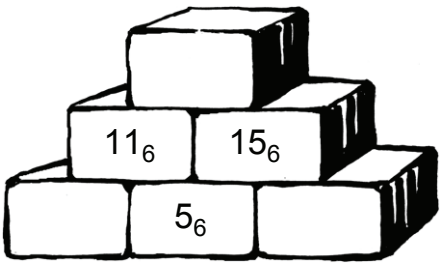
2. Number walls

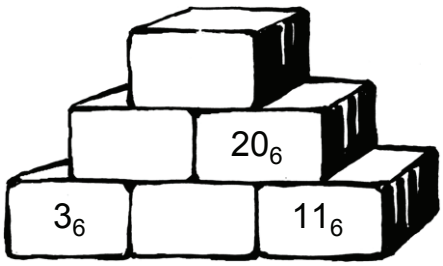












Name: Date:

From a School Book in Dream Land

3. Calculate. (You can use any materials in order to help you.)

12_6	35_6	24_6	103_6	444_6
$+\ 13_6$	$+\ 12_6$	$+\ 44_6$	$+\ 54_6$	$+\ 222_6$
23_6	32_6	41_6	100_6	1111_6
$-\ 11_6$	$-\ 14_6$	$-\ 32_6$	$-\ 55_6$	$-\ 300_6$

4. Find the multiples of 5 in base-six. What do you notice?

5. Find the square numbers in base-six. What do you notice?

6. What is the largest four-digit number and what is the smallest four-digit number in the base-six system?

7. How many two-digit, three-digit, four-digit, ... numbers are there in the base-six system?

7 Arranging High-Speed Trains

Combining and Calculating in Real-life Situations

K. Weigl

Topic and Purpose

In the context of a familiar situation – travelling by high-speed train – the pupils can use their individual experiences and develop their calculating skills. They can practice the different basic calculations, the operations depending on their abilities. Besides doing exercises and practicing, mathematical connections based on concrete situations become accessible for the pupils. Initially, it seems the pupils are practicing calculating. But, upon taking a closer look at the exercises, the pupils focus on systematical and mathematical connections. It is no longer just calculating.

With regards to content, this section is about examining high-speed trains, which the children know from everyday life, and finding different possibilities for arranging the rail cars. In order to calculate as many different combinations as possible, the pupils have to combine the rail cars, the number of seats and the prices.

Background

Most high-speed trains consist of first- and second-class rail cars and the dining car. The facilities, the transport capacities and the prices for various trains differ. The lengths of the rail cars can be added up to find the length of the whole train, or the length can be found using multiplication. For example, the pupils can calculate how many seats a train with 2 first class cars and 6 second class cars has. Its length can be easily calculated. It gets more difficult when the pupils should calculate the length of a train with 400 seats. There are different ways that lead to different solutions. Other parameters concerning train calculations could be ticket prices and the maximum revenue of a train. Another idea for further practice is creating a train that is as short as possible regarding certain conditions. The pupils become familiar with those different basic conditions and use one or a combination of them, according to their individual skills. These types of exercises offer the possibility for each child to work on this topic according to his or her level of mathematic skills.

Of course it is also possible to define clear criteria for each level of mathematical proficiency such that the teacher can evaluate the pupils' assignments clearly and fairly. The pupils can correct the exercises by themselves, or the teacher can check them by discussing the definition of the problem, the first approach, special insights and distinct solutions in class. Always using "can-phrases" in the exercises shows the pupils that there is not a unique solution but that there are several ways of finding different results.

Methodological Advice

The pupils learn basic information about high-speed trains and become aware of different possibilities for easy calculations related to this topic (see above). Concrete definitions of the problems and discussing the approach will help to ensure that every child understands the problems. Afterwards, the pupils are asked to arrange the train cars according to the special characteristics of the train. In order to stimulate the pupils' imagination, some possibilities are already given. These suggestions do not have to be used by all children. Each child should be able to arrange the train cars based on the given information. Furthermore, the trains should have special characteristics that the children are able to describe and work on mathematically. Of course, it is the teacher's responsibility to help the pupils if they have problems with arranging the train cars or choosing certain characteristics of the train.

Name: Date:

The High-Speed Train – One Train, Many Possibilities

At first sight, all high-speed trains look the same. But in reality, there are a lot of possibilities on how the different train cars can be arranged.
A high-speed train consists of first class train cars, each of which can seat 50 people, and second class train cars, each of which can seat 70 people. Between the first class and second class train cars, there is the dining car with the on-board restaurant. Every rail car is 25 meters long.

1. How many seats can a train with 5 rail cars have? Find several solutions.

2. How long can a train with 400 seats be?

3. How many rail cars can a train have if the numbers of first class and second class seats are the same?

4. From A-Town to B-Village, a first class ticket costs 100 Euro, and a second-class ticket costs 75 Euro.
   How much money can the train company earn if the train has 5 rail cars, both classes included?

5. Arrange your own high-speed trains, which have certain characteristics that you choose.

   Examples:
   a) *about 600 seats; half the number of first class than second class*

   b) *high earnings possible with 6 rail cars, both classes included*

   c) *a short train; high earnings; for about 500 persons*

   Please note: The longer the train is, the more expensive energy it needs. Also, seats in first class are more expensive, and thus the train company earns more money.

8 Placing Marbles into Bags

Finding all Possibilities by Systematic Experimentation

J. Schoft, T. Sinning, A. Stix-Pöhner

Topic and Purpose

In this section, the pupils are confronted with problems from combinatorics. They combine colored marbles in a bag. Because the children have real marbles, they can try different combinations. In the beginning, most of the children usually make discoveries unsystematically by playing with the marbles. Given some time, the pupils will think of structures that help them to plan their experiments and thus to find as many solutions as possible. The knowledge gained here can be used later for similar problems.

Background

When putting the marbles into the bag, each color can occur more than once. The order of the marbles is not important, because they will be mixed up in the bag. A possible solution for structuring the problem is to consider the number of colors in each bag. Afterwards, all partial results are added up. You find the following solutions for each of the five exercises:

Tim: 2 possibilities if the marbles in the bag have the same color + 1 possibility if they have two different colors = 3 possibilities

Oliver: 3 possibilities if the marbles in the bag have the same color + 3 possibilities if they have two different colors = 6 possibilities

Laura: 3 possibilities if the marbles in the bag have the same color + 6 possibilities if they have two different colors + 1 possibility if they have three different colors = 10 possibilities

Sue: 4 possibilities if the marbles in the bag have the same color + 12 possibilities if they have two different colors + 4 possibilities if they have three different colors = 20 possibilities

Lizzi: 3 possibilities if the marbles in the bag have the same color + 6 possibilities if they have two different colors at a ratio of three to one + 3 possibilities if they have two different colors at the ratio of two to two + 3 possibilities if they have three different colors = 15 possibilities

The most difficult part of the problems is finding the possibilities if the marbles in Laura's, Sue's and Lizzi's bag have two different colors. A tree diagram, which shows all possible outcomes of an event, can help.
The general formula for the total number of possible marble bags is quite complicated. If you have n colors and k marbles per bag, there will be

$$\frac{(n+k-1)!}{(n-1)!\cdot k!}$$

possibilities. The exclamation mark means factorial. For example: $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

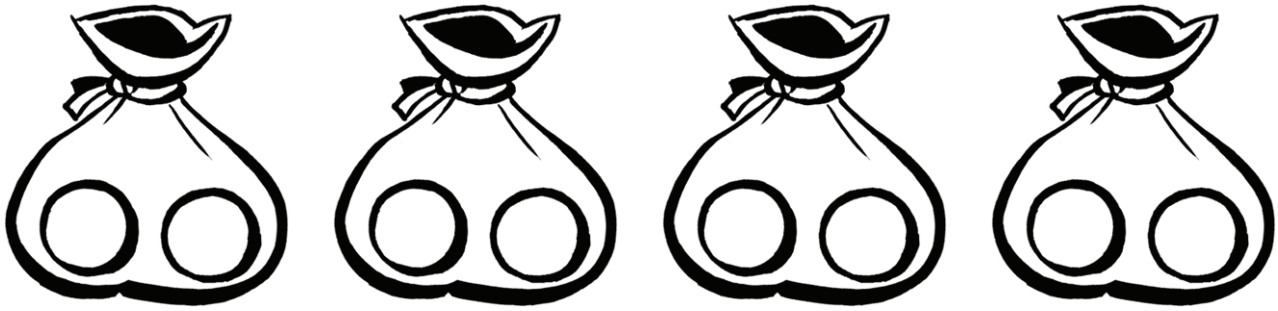
Methodological Advice

The pupils should be able to combine and try out different possibilities with actual marbles. In the beginning, they work on the first problem together sitting in a circle, on the floor or in chairs. The marbles help them to find different possibilities, which can be discussed and later written on the board. The pupils should understand that the marbles can be mixed up in the bag and therefore combinations such as "red-blue" and "blue-red" are the same. Then, the pupils should develop their own strategies to find solutions. The assignments used should depend on their level of proficiency.
To enhance the learning effect, it is recommended to alternate between working individually, discussing in small groups and reflecting in class. This topic is mainly aimed at planning experiments and grasping structures rather than finding all possible solutions.

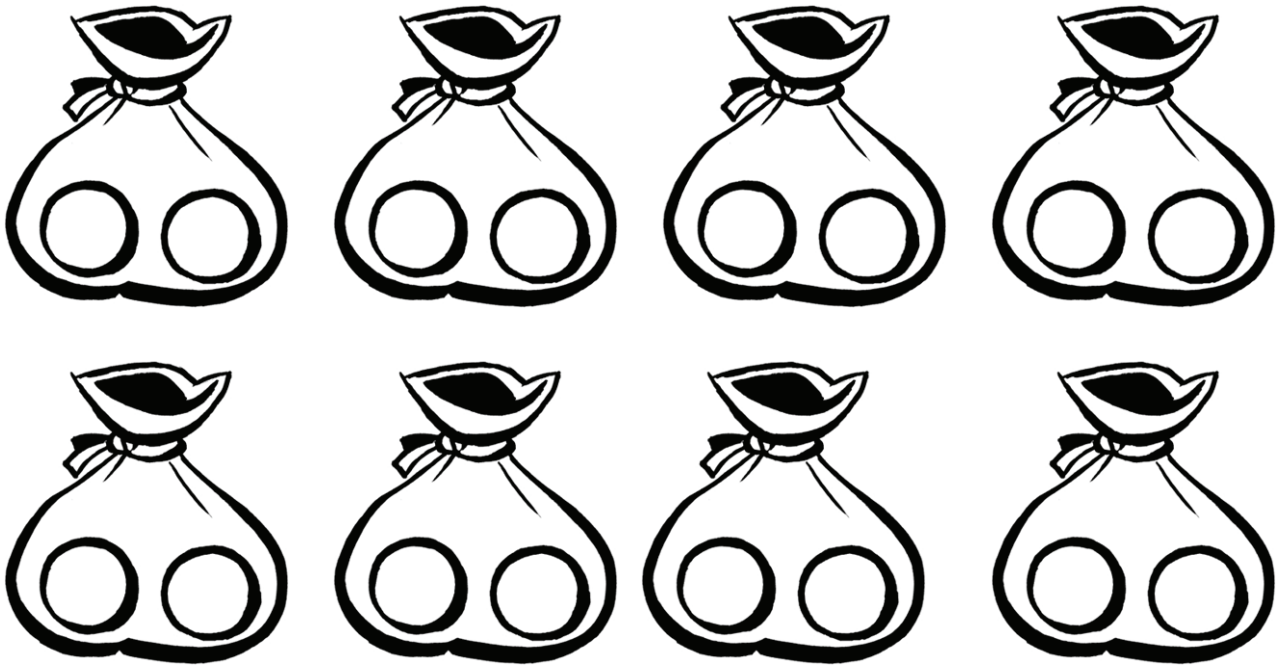
Name: Date:

Bags with two marbles

Tim has red and blue marbles.
He puts 2 marbles into his bag.
Which combinations of colors can occur? Find all possibilities and color in the marbles!



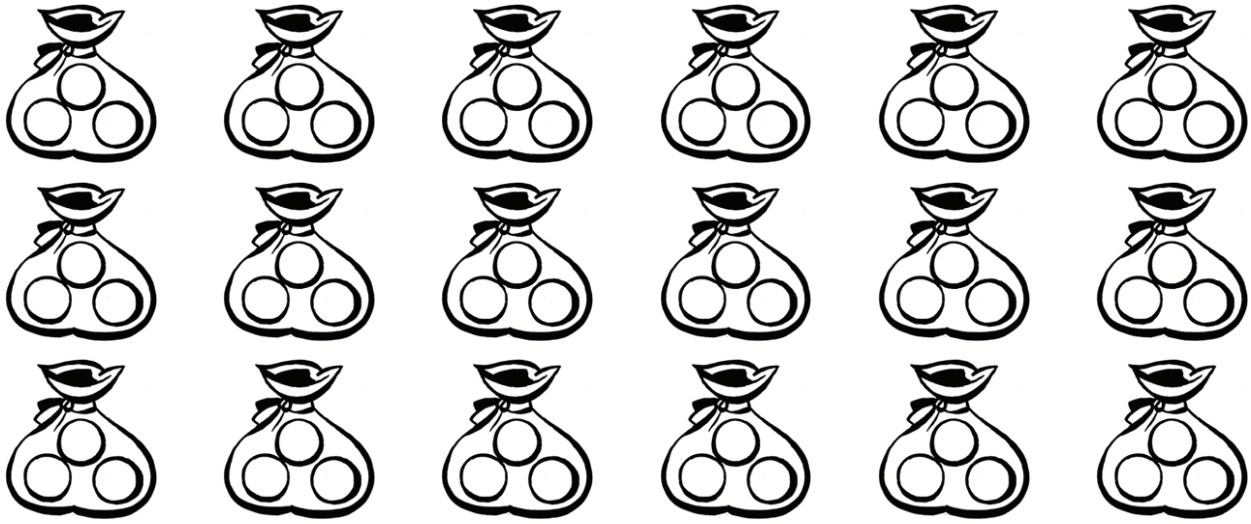
Oliver has red, blue and yellow marbles.
He puts 2 marbles into his bag.
Which combinations of colors can occur? Find all possibilities and color in the marbles!



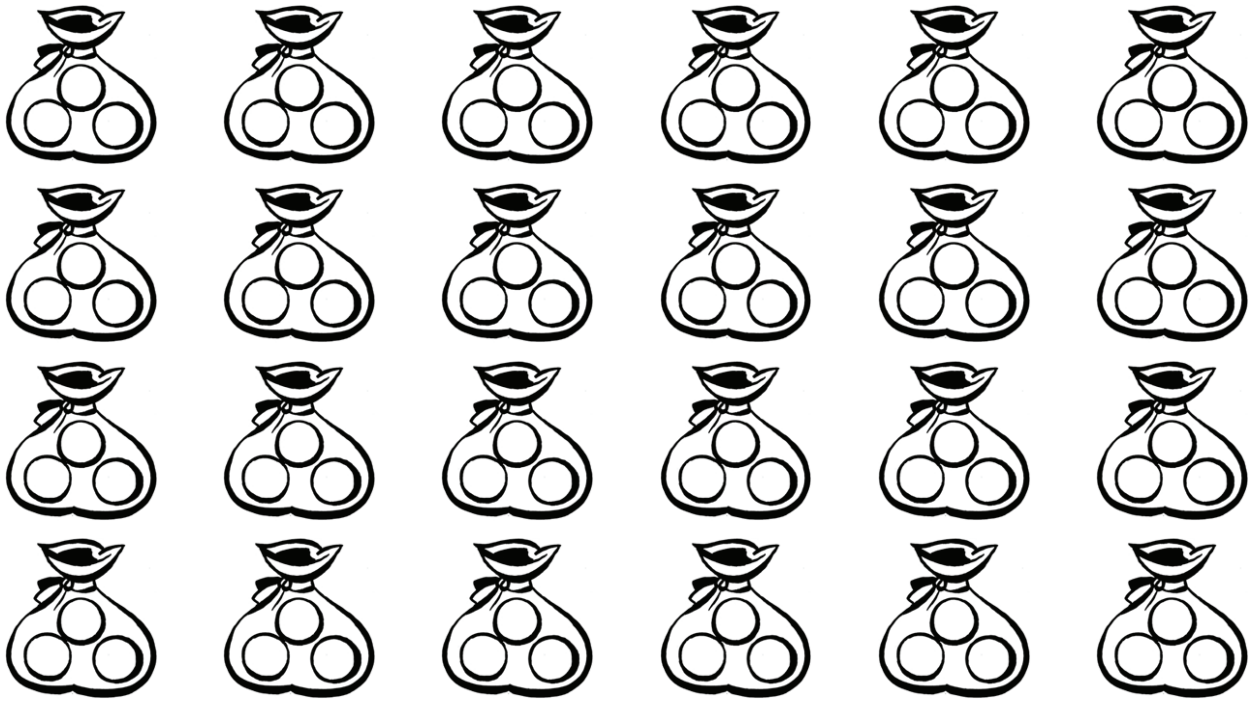
Name: Date:

Bags with three marbles

Laura has red, blue and yellow marbles.
She puts 3 marbles into her bag.
Which combinations of colors can occur? Find all possibilities and color in the marbles!



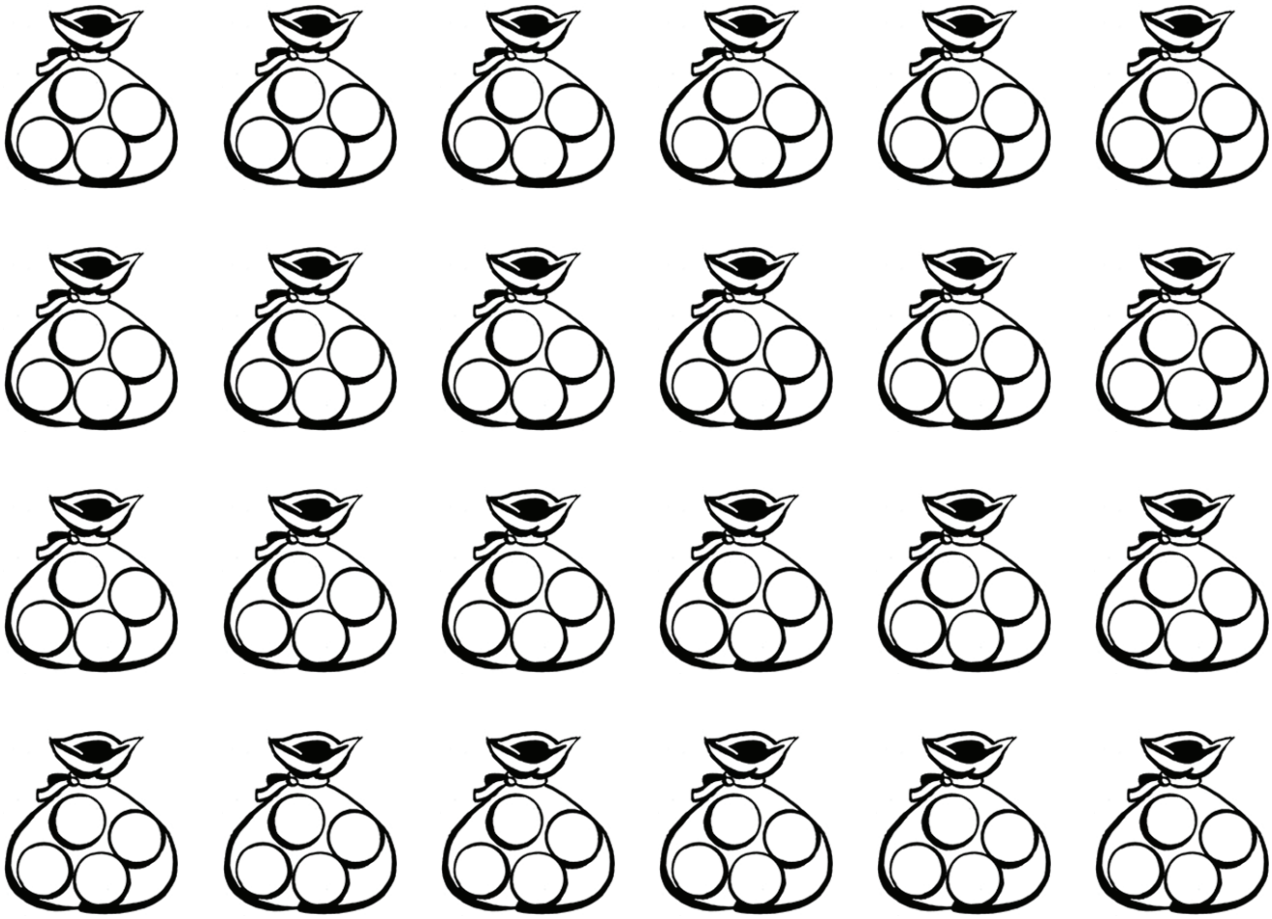
Sue has red, blue, yellow and green marbles.
She puts 3 marbles into her bag.
Which combinations of colors can occur? Find all possibilities and color in the marbles!



Name: Date:

Bags with four marbles

Lizzi has red, blue and yellow marbles.
She puts 4 marbles into her bag.
Which combinations of colors can occur? Find all possibilities and color in the marbles!



9 Secret Languages

Algorithmic Thinking through Cryptography

V. Ulm

Topic and Purpose

Secret languages fascinate children. They can exchange secret messages that other people such as adults cannot understand. In this section, the pupils can explore three different coding techniques. They encrypt and decrypt their own messages and explain the principles on which the techniques are based. This process supports the children's abilities to think algorithmically. Furthermore, the pupils learn that they have to use the coding algorithm precisely to get the original text back.

Background

Cryptography occurs in everyday life, but it is not obvious at first sight. With phone cards, remote controls, mobile phones, cash machines (ATMs), pay-TV and immobilizers in keys for cars, you find complicated structures for encrypting and decrypting information.

The following rather easy examples are based on two different principles of encryption. The principle of *substitution* changes every letter of a text into another symbol. This symbol takes the same place as the original letter did. For both encryption and decryption, you need a reversible rule to find the right symbol for each letter. Caesar's technique is based on this principle: Every letter of the alphabet is assigned to another letter by counting forward, and accordingly the secret text is decrypted by counting backward.

Using the principle of *transposition*, all letters remain the same, but they are put in a different order. You need an invertible rule that tells the new position of every letter of the text. Examples include the garden-fence and stencil encryptions.

Methodological Advice

The introducing worksheet *Encrypting like Caesar* contains a lot of text because the pupils should work on this topic independently – either individually, in pairs or in small groups. Thus, they learn how to extract complex information out of a text. The teacher should ensure that the children understand the rules of the structure of the encryption after they have worked on the first exercise.

The Caesar-disks should not be used from the very beginning. First, it is important for the pupils to grasp the idea of counting forward and backward within the alphabet. After a while, the Caesar-disks can make the encryption and decryption easier. The pupils should figure out how encoding and decoding works on their own and learn to describe the process independently.

The garden-fence and stencil encryptions are presented by just one example each. It is the pupils' task to find out the general principle of encryption, to explain it and to think of their own examples.

The stencil encryption supports geometrical thinking and deepens understanding of rotational symmetry. The stencil should show different parts of the inner grid every time it is turned. Because there are 36 small fields in the grid, not more than 9 fields can be cut out. Moreover, several fields will lie on top of each other after rotating the square by 90°, 180° or 270°. Only one of these coinciding fields should be cut out in the stencil.

Further Reading

Beutelspacher, A. (2005): Geheimsprachen, Beck, München

Beutelspacher, A. (2010): Kryptologie, Vieweg, Braunschweig

Name: Date:

Encrypting like Caesar

Secret languages encrypt texts so that unauthorized persons are not able to read them. Only the intended addressee should know how to decrypt the text.

One of the oldest ways of encryption comes from the Roman statesman and general, Gaius Julius Caesar (100 – 44 BC).
Imagine all letters of the alphabet in one row:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

We choose the encrypting number 5. This means that you have to move 5 fields forward for encrypting a letter. So, A becomes F, E becomes J, and S becomes X.
As soon as you come to the end of the alphabet, you start from the beginning. So, W becomes B, and Y becomes D.

That way, you can encrypt every word, letter by letter. The word “WATER” becomes the secret word “BFYJW”. This secret word cannot be understood unless the encrypting number 5 is known. To decrypt the word, you have to move 5 fields backwards from each letter: B becomes W, F becomes A, etc.

Instead of using 5 as the encrypting number, you can use any other number, such as 2 or 12. Thus, other secret words will result.
The encrypting number is a secret. You should only tell someone the number if you want him or her to be able to decrypt your text.

1. Choose other encrypting numbers and encrypt words.

2. What does the following text say? (The encrypting number is 7.)

   AOPZ ALEA PZ AVW ZLJYLA

3. Choose an encrypting number together with your partner and write secret messages to each other the way Caesar did it.

4. Can you find out the following code without knowing the encrypting number? Explain.

   DRO YXO GRY MKX BOKN DRSC SC MVOFOB

Name: Date:

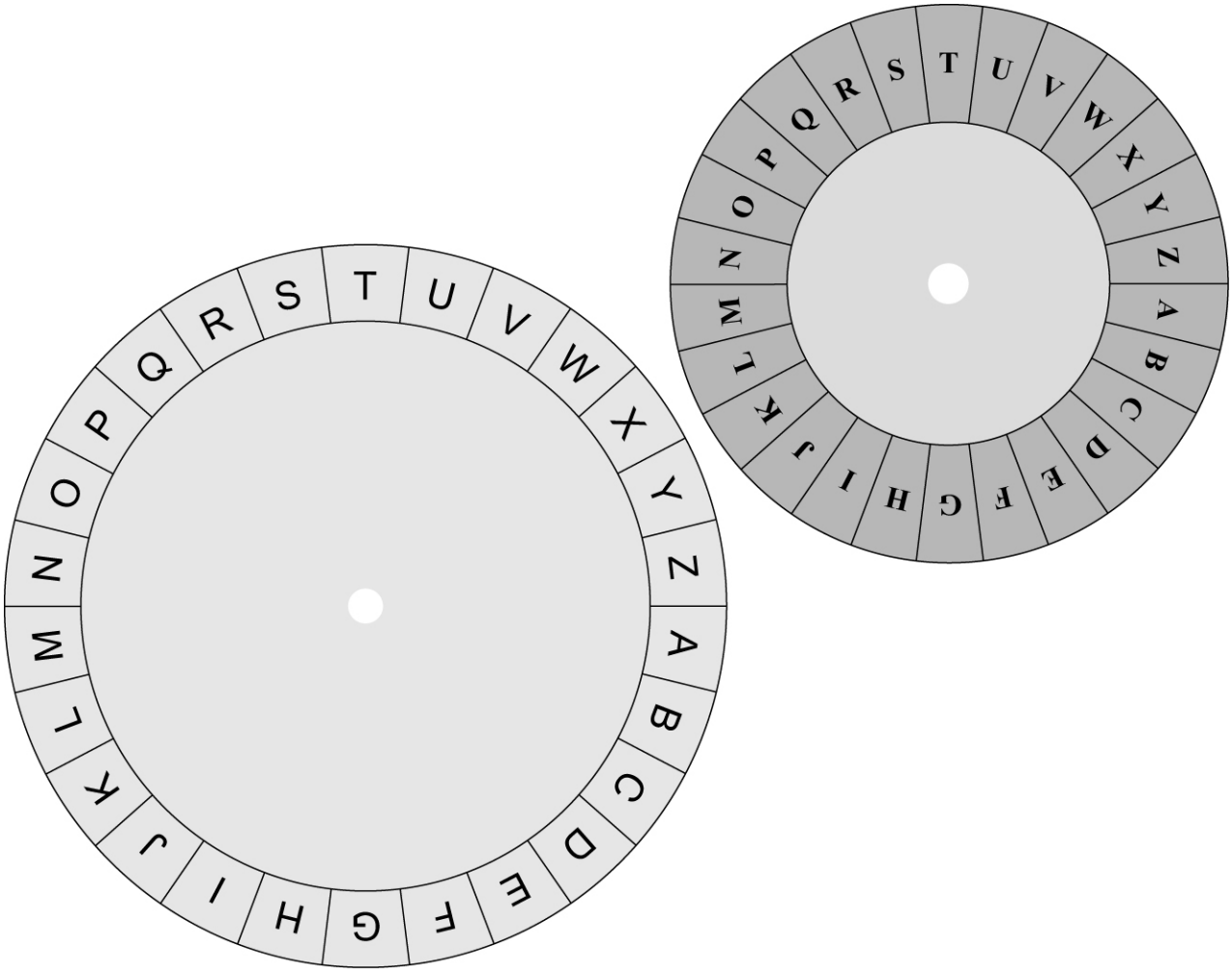
Caesar-Disks

So far, you have needed a lot of time for counting forward and backward. There is an easier way:

1. Cut out both disks. Fix the smaller disk on the bigger one by connecting them with a clip through the hole in the middle, so you can turn both disks.

2. On one disk, there are the letters of the text which is not encrypted. On the other disk, there are the letters of the secret text.
   Explain how to encrypt and decrypt texts with these disks.

3. Use the Caesar-disks to exchange secret messages with your schoolmates.



Name: Date:

Garden Fence Encryption

It is difficult to understand the following secret text:

IIOMRARIVTYUOYITDYATNETBHPY

It comes from the following original text:

I				I				O				M				R				A				R		
	I		V		T		Y		U		O		Y		I		T		D		Y		A		T	
		N				E				T				B				H				P				Y

1. Explain how you can encrypt and decrypt the message.

2. Exchange messages with your schoolmates by using the so called “garden fence encryption”.

Stencil Encryption

The text in the grid shown below is hard to understand.

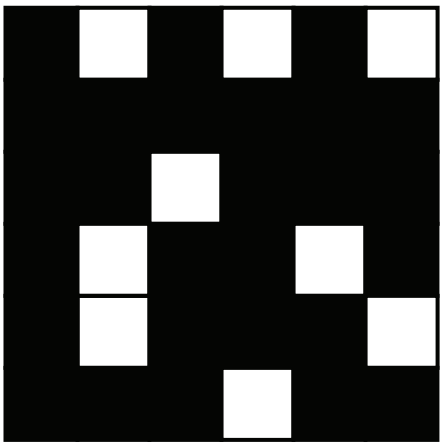
1. Cut out the stencil on the right, and cut out all the white areas.

2. Put the stencil on the grid. Write down what you can read.
   Turn the stencil clockwise. Write down the message after every quarter turn.

3. Explain how to encrypt and decrypt texts with this stencil.

4. Encrypt messages for your schoolmates.
   Find other stencils. What do you have to consider?

F	L	T	E	T	T
O	I	N	E	M	T
R	O	‘	H	R	N
E	S	O	R	M	P
O	E	A	N	.	E
O	R	W	T	A	K



10 Fermi Problems

Investigating, Estimating, Calculating, Interpreting, Discussing

B. Adleff

Topic and Purpose

The Italian physicist and Nobel laureate Enrico Fermi (1901 – 1954) is known for the special kind of questions he posed. These questions should help his students to solve problems, which seem unsolvable, using common sense. A Fermi question typically involves a simple, possibly surprising problem out of everyday life. However, the problem appears impossible to compute due to limited available information. Such problems require estimating, collecting information, approximating, comparing and discussing to approach possible solutions. Therefore, Fermi problems are challenging.

Background

Fermi questions describe problems in the context of the children's world, in a way that is accessible to the pupils. The questions are posed clearly and openly. Due to limited available numerical information, the pupils are required to collect information and to estimate. They must think of sub-questions that they can answer. In this way, they can approach the solution of the initial problem. It is not important – and mostly it is not even possible – to find one unique solution. This topic rather aims at grasping the idea of the order of magnitude by making justified guesses and appropriate estimations.

Such mathematical activities help to develop the pupils' mathematical skills, because the children:

- become aware of mathematical problems in everyday life and practice their abilities in mathematical modeling,
- are encouraged to be courageous and self-confident as they use their problem-solving skills for finding solutions for problems which seem to be unsolvable,
- deepen their understanding of orders of magnitude and connect their ideas of everyday life with mathematics,
- do precise and rough calculations with numbers and units of measure,
- get used to estimations and working with limited available information,
- learn how to reflect and evaluate their approach, and realize that different assumptions and estimations can lead to equally "good" solutions.

Methodological Advice

The worksheet provides some questions that are directed at 4th grade pupils. Some Fermi problems can be posed directly to younger pupils by writing them on the board. Moreover, you can make the problems easier, adjusting the numbers to ones the children have already learned, by replacing "pupils in our school" by "pupils in our class," or "one year" by "one day" or "one week".

It is recommended to alternate between working individually, in pairs or in small groups and discussing the ideas in class – in particular, if the children work on different problems.

Further Reading

Büchter, A. u. a. (2007): Die Fermi-Box, Friedrich Verlag, Seelze

Name: Date:

Fermi Problems

Choose one of the following questions with your partner, and try to answer it. Present your results in a clearly structured fashion.

1. How many meters of spaghetti could all the children in our class eat during lunch together?

2. Do all the children in our school fit into our classroom?

3. How often does the letter “e” occur in a thick book?

4. How many times in your life has your heart beaten until now?

5. How many soccer balls fit on our sports field?

6. How much air have you breathed in your life so far?

7. How much toilet paper do all the people in our school use per year?

8. If all pupils in our school pile up their school books, how tall would the tower of books be?

9. All cars in town are parked in a line. How long is this line?

10. How many packages of drinks would we need for an event at our school?

11. How much time in your life have you slept until now?

12. How many kilometers have you gone on foot until now?

13. How much drinking water do you and your family need per year at home?

14. How many inflated balloons fit into our classroom?

15. How much time in your life have you watched TV up to now?

16. How many gummy bears fit into a school bus?

Create similar questions and try to answer them.

11 Paper Formats and Origami

Paper Folding and Discovering Relationships between Lengths

I. Weigand

Topic and Purpose

In this section, the pupils fold paper pockets of different sizes. They measure these pockets and discover connections between their measurements. Thus, geometrical and numerical thinking are combined and supported. DIN-A formats provide familiar geometrical shapes from everyday life through which the children can make mathematical discoveries. For the purposes of cumulative learning, the children combine everyday life with mathematics (see Wittmann, Müller, 2007, p. 60, and also Mulatinho, 1996, p. 9).

Background

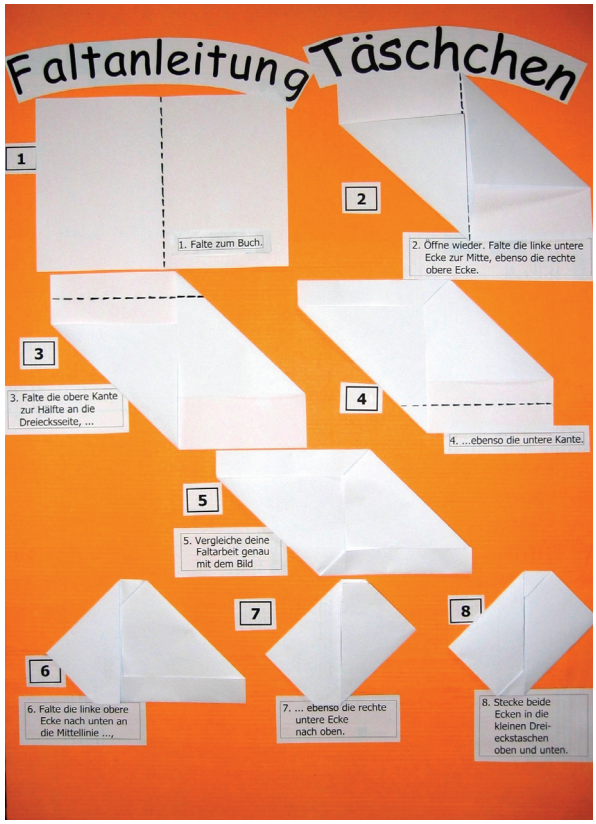
A paper in a DIN-A format can be turned into the next smaller DIN-A format by bisecting the original. Furthermore, all DIN-A formats are similar. Consequently, the ratio of length and width of a DIN-A piece of paper is $\sqrt{2}$, which means that the length is about 1.41 times as long as the width. The surface area of an A0 piece of paper is 1 m². The following table shows the dimensions of DIN-A formats:

	length	width
(DIN-)A0	118,9 cm	84,1 cm
(DIN-)A1	84,1 cm	59,5 cm
(DIN-)A2	59,5 cm	42,0 cm
(DIN-)A3	42,0 cm	29,7 cm
(DIN-)A4	29,7 cm	21,0 cm
(DIN-)A5	21,0 cm	14,9 cm
(DIN-)A6	14,9 cm	10,5 cm

If you fold a pocket out of a A4 piece of paper according to the folding instructions, the pocket will be as big as an A6 piece of paper. The pocket folded out of an A5 piece of paper is as big as an A7 piece of paper, etc. So, every pocket is half as long and half as wide as the used folding paper.

Methodological Advice

The pupils get the folding instructions on a worksheet or on a poster (see picture). Standard A4 paper is suitable for folding, e.g. coloured copying paper.
The pupils should get all the measurements from their own folding. They should describe the main observations by using the words “double”, “half” and “bisect”.
In the end, all pockets can be put into order according to their sizes and stuck on a poster. The pupils can write down their measurements in a table on the poster.

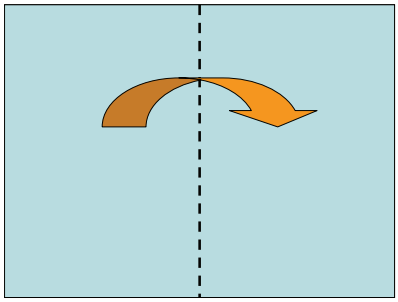


Further Reading

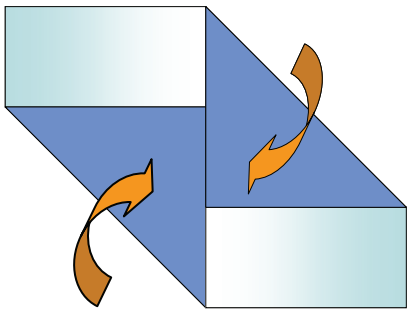
Mulatinho, P. (1996): Pfiffiges Origami, Augustus Verlag, Augsburg
Wittmann, E., Müller, G. (2007): Das Zahlenbuch 3, Bayern, Klett, Stuttgart

Name: Date:

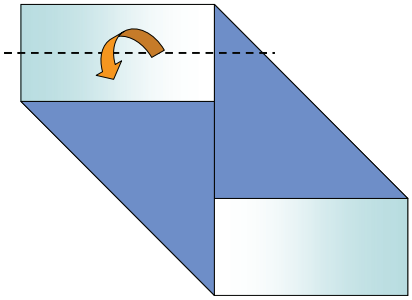
Folding Instructions: The Origami Pocket



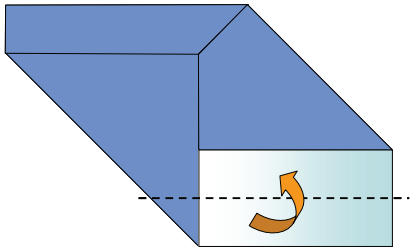
1. Fold in half.



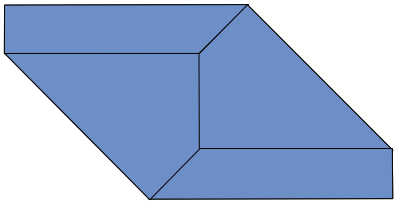
2. Reopen. Fold the bottom left corner to the middle, do the same with the right corner.



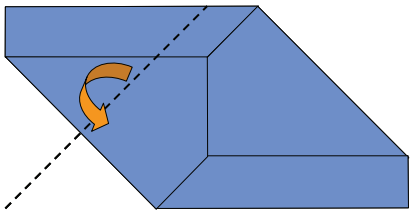
3. Fold the top edge in half, up to the edge of the triangle, and…



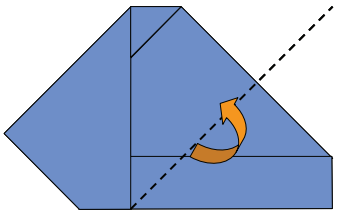
4. ... do the same with the bottom edge.



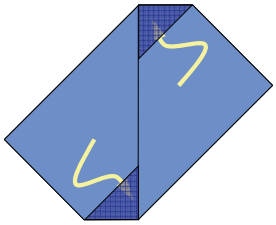
5. Compare your work with the picture.



6. Fold the top left corner down to the middle line, and...



7. ... fold the bottom right corner up to the middle line.



8. Place both corners into the triangular shaped pockets.

Your pocket is done!

Name: Date:

Report on Folding: My Pocket

1. What did you pay attention to while folding your pocket?

2. Which number in the folding instructions is the most difficult? Why?

3. What could you put into your pocket?

4. How did you succeed in folding your pocket?

Exercises on Measuring and Drawing

1. Measure the dimensions of a second folding paper and your pocket. Fill in the table:

	Length	**Width**
Folding paper		
Pocket		

   Compare the measurements of the folding paper with the ones of the pocket. What do you notice?

2. Sketch a drawing of the pocket (front side and back side) on the back of this worksheet.

Name: Date:

The Folded Pocket Becomes Smaller and Smaller

First measure the folding paper, then the pocket.
Write your measurements in the table.

	Length	Width	Exercise
Folding paper 1			Start with an A4 piece of paper, measure and fold.
Pocket 1			
Folding paper 2			Fold and cut an A4 piece of paper in half, measure and fold a new pocket.
Pocket 2			
Folding paper 3			Fold and cut the rest of this piece of paper in half, measure and fold again.
Pocket 3			
Folding paper 4			Fold and cut the new rest of this piece of paper in half, measure and fold again.
Pocket 4			
Folding paper 5			Fold and cut the rest of this piece of paper again in half, measure and fold.
Pocket 5			
Folding paper 6			Can you also fold the smallest form of the pocket?
Pocket 6			

Compare your measurements. What do you notice? Explain.

12 Geometry through Paper Folding: The Origami Fox

Supporting Geometrical Thinking through Origami

I. Weigand

Topic and Purpose

All children know the clever fox. He appears in many stories such as *Tomte and the Fox* (A. Lindgren), in any tale where the fox is a character of a fable and as a wild animal in our forests. You can connect this fictional clever fox with the paper folding *origami fox,* which is presented in this section.

Paper folding geometry is highly motivating for pupils, but it also encourages developing motor skills, working precisely and thinking in a structured and geometric way. Pupils easily succeed in paper folding and get a handle on basic concepts of geometry. Geometric skills can be gained by transferring between plane and space.

Background

With origami, the pupils have to recognize structures, read texts, understand the steps of the paper folding, carry out these instructions and reflect on their results through the paper folding report. Several different geometric shapes result from the paper folding: triangles, squares, kites, trapezoids, general quadrilaterals. These shapes are mostly symmetric. Thus, geometry is discovered in a complex but interesting way.

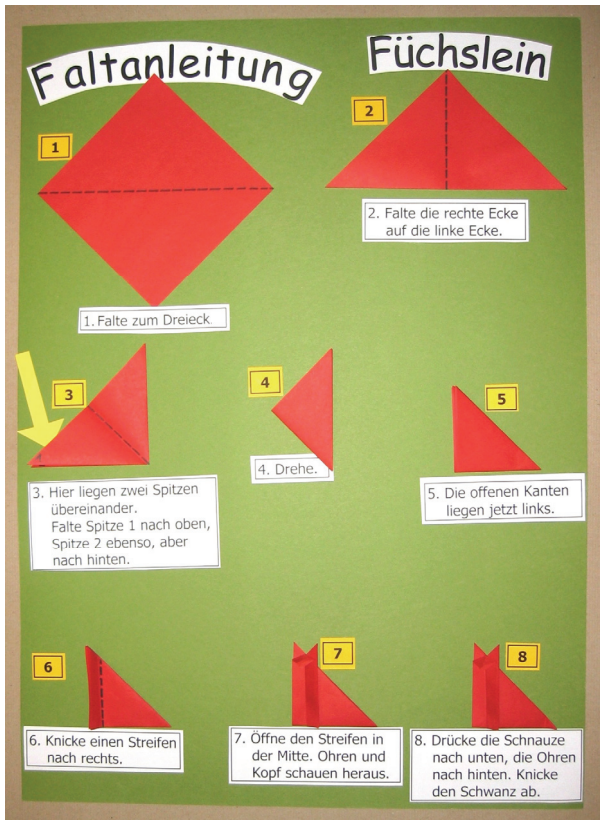
The worksheet *Exercises on Animals' Legs* continues this topic. It focuses on arithmetic and problem solving. First, the pupils calculate the multiples of four; then they solve rather simple problems that are based on addition. In the following exercises, the value of the sum (number of the animals' legs) is given. The pupils need to develop problem-solving strategies and experiment with different possibilities systematically by drawing and using different material.

Methodological Advice

The pupils get the paper folding instructions on a worksheet or a poster (see the picture). For folding they need a square piece of paper which is between 15 to 20 cm long per side.

The lesson can be structured as follows:

- Look at an example; examine its shape and symmetry.
- Fold the fox independently using the folding instructions.
- Present all the origami foxes.
- Reflect on the folding process (report on difficulties, give tips, etc.).
- Work on the paper folding report and fill in the shapes chart.
- Unfold the fox and study in detail the unfolded paper; recognize certain shapes.

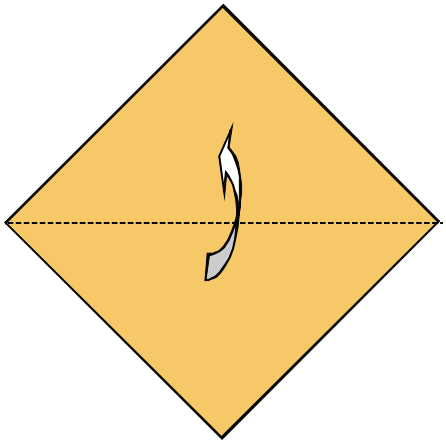


Further Reading

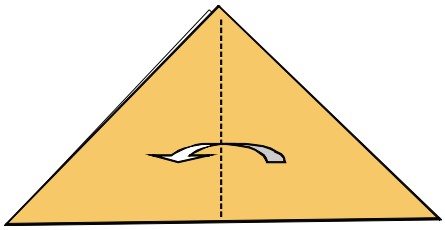
The folding instructions are based on: www.basteln-gestalten.de/fuchs-basteln

Name: Date:

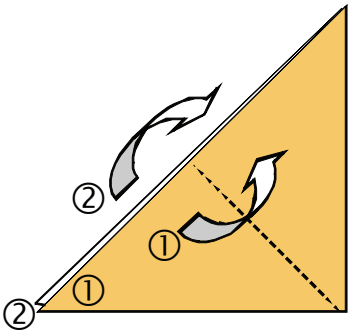
Folding Instructions: The Origami Fox



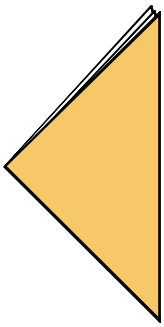
1. Fold into a triangle.



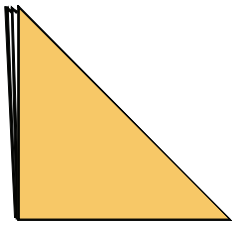
2. Match the right corner, called tip 1, with the left corner, called tip 2.



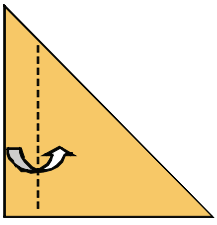
3. Here the two tips are aligned. Fold tip 1 forwards and upwards and fold tip 2 backwards and upwards.



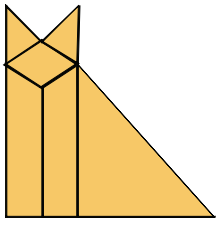
4. Turn counter-clockwise.



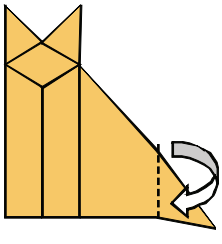
5. The open edges are on the left side.



6. Fold one strip to the right.



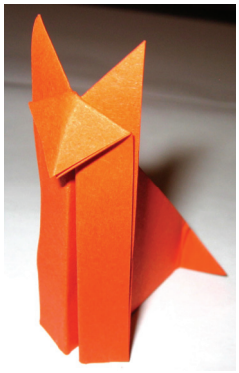
7. Open the strip in the middle. You can now see the fox's ears and head.



8. Press his nose downwards. Fold his tail.

Name: Date:

Paper Folding Report: The Origami Fox



1. Look at your origami fox in detail. What do you discover?

2. What did you pay attention to while folding the fox?

3. Which was the most difficult part?

4. How did you succeed in the paper folding?

5. In the paper folding instructions, you can discover many different shapes. Which ones? How many are there?

	Triangles	Squares	Other quadrilaterals
Picture 1			
Picture 2			
Picture 6			
Picture 8			

Name: Date:

Exercises on Animals' Legs

1. How many legs do 1, 2, 3, … foxes have? Write it down and describe the sequence of numbers.

Foxes	1	2	3	4	5	6	7	8
Legs								

2. A hungry fox sneaks around the chicken yard and finds three chickens. How many legs do the fox and chickens have all together?



3. Now there are 3 chickens, 1 fox, 2 ducks and 4 geese. How many legs do the animals have all together? Draw and calculate.

4. There are 4 chickens, 3 ducks, 2 geese, 2 goats and the fox, who is still sneaking around. How many legs do the animals have all together?

5. Some animals flee into the barn. The fox is still waiting in the yard. All animals in the yard together have 16 legs. Which animals could still be in the yard? Think of different possibilities.

6. In the yard, there are twice as many chickens as ducks and twice as many ducks as goats. The fox is still there too. All animals together have 36 legs. How many animals are in the yard?

7. Create your own chicken yard exercises.

13 Brain-Teasers with Pentacubes
Encouraging Spatial Perception

B. Brandl, M. Brandl

Topic and Purpose

Many pupils may be familiar with the Soma cube from mathematics lessons. Its parts consist of three or four cubes with the same edge length. Pentacubes consist of five identical cubes and have 29 different combinations.

Working with pentacubes develops spatial perception, a skill which does not always occur naturally. Up to the ages of 9 and 10, about 50 % of the ability of spatial perception has been developed, and up to the ages of 12 to 14 about 80% of the ability of spatial perception is available – with percentages referring to the average spatial perception of an adult. Spatial perception develops during primary school, and because it can be learned, the teaching of spatial perception in mathematics lessons is both rational and important.

Moreover, surveys confirm that training with a practical focus and the use of models and media is rather effective. Therefore, the lessons should have a practical focus.

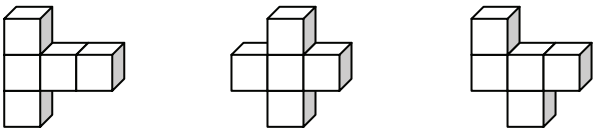
At first, the pupils build as many different pentacubes as possible using unit cubes. Afterwards, the children put the pentacubes into order according to their own criteria. Then they make oblique drawings of pentacubes based on photos and construction plans and find mistakes in given oblique drawings. They also build combinations of pentacubes according to given side views. The game *Pentacube Quartet* encourages and supports spatial perception, because the children have to imagine how to turn and rotate the pentacubes.

Background

1. Who Can Find All 29 Pentacubes?

Whereas the possible combinations of three or four identical cubes can be found out easily, the pupils need to find a strategy to build pentacubes. This strategy helps them maintain an overview of the number of different pentacubes. If the pupils first do not develop a system, they will need to develop one during the second part of the task, which is about classifying the pentacubes into categories. Possible categories can be determined by the longest straight chain of cubes within the pentacube or by the similarity to Arabic digits.

On the worksheet *Pentacube Overview*, the pentacubes labeled with the tens 10, 20, 30 etc. are similar to the numbers one, two, three, etc. The pentacubes that follow to the objects with whole tens come from the "tens pentacubes" by simple changes (see Künzell 1995, p. 5).



Chains of 3 cubes



Chains of 4 cubes

Regarding the symmetry of the pentacubes, three categories are found:

- Pairs of pentacubes which are symmetrical to each other
- Pentacubes which have at least one plane of symmetry and are symmetrical to themselves
- Pentacubes without any symmetry

The solution, using the same numbering as on the worksheet *Pentacube Overview*, is:

- Category 1: 21-22, 31-32, 33-34, 35-36, 41-42, 71-72
- Category 2: 10, 11, 12, 13, 20, 30, 37, 40, 50, 51, 60, 70, 80, 81, 82, 90
- Category 3: 61

Furthermore, the pupils should wonder if each pentacube belongs to only one category and which pentacube is the most symmetrical.

2. Building with Pentacubes

When creating buildings with pentacubes based on given side views, the pupils have to be creative, utilize good spatial perception and transform two-dimensional drawings into three-dimensional objects. A square grid, which contains squares as big as the edge of the cube and on which the four directions are noted, can be helpful.

Of course, the pupils can even be more creative and construct their own shapes with the pentacubes, and draw the side views and the bird's-eye views. Are the other pupils able to build these objects?

3. Drawing Pentacubes: Making Two Dimensions Out of Three

This topic deepens understanding of oblique views. The pupils can draw on graph paper or on paper with a grid. The teacher can decide how accurately the pupils should draw and which rules of oblique drawing should be followed. Furthermore, finding mistakes in drawings is like creating a drawing in reverse. This process helps the children reflect on what they have learned. Usually, pupils like to find mistakes and correct them.

4. Pentacube Quartet

Players of the Pentacube Quartet have to collect four cards which show the same object. The pentacubes are shown from different perspectives. The rules are equal to the well-known rules of ordinary Quartet. It is recommended that the teacher provides pentacubes for the pupils to check the solution.

Advice:

- On the copy, there are four cards that belong together in a row.
- It is recommended to enlarge the copy (A3 piece of paper) so that the children hold the cards more easily in their hands. Also, the card will last longer if they are laminated.

Methodological Advice

The pupils should work on the first two activities either in pairs or in small groups, because the tasks are quite difficult. They can later play Pentacube Quartet in groups of four.

For the section *Drawing Pentacubes: Making Two Dimensions Out of Three*, the pupils should have a set of the 29 different pentacubes. This material can be helpful for other topics, too. Such a set of pentacubes can be made out of unit cubes by the pupils themselves during the activity *Who can find all 29 Pentacubes?* Alternatively, either the teacher or the class can glue wooden cubes together to create pentacubes in preparation for this topic.

Further Reading

Carniel, D. (2007): Räumliches Denken fördern, Auer, Donauwörth

Franke, M. (2001): Didaktik der Geometrie, Spektrum Akademischer Verlag, Heidelberg

Künzell, E. (1995): Spiele mit Pentakuben, Aachen, siehe auch: http://www.pentakuben.de

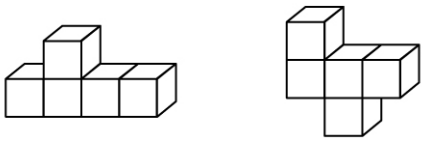
Maier, P. (1999): Räumliches Vorstellungsvermögen, Ein theoretischer Abriß des Phänomens räumliches Vorstellungsvermögen, Auer, Donauwörth

Name: Date:

Who Can Find All 29 Pentacubes?

Pentacubes consist of five identical cubes which form an object. The name derives from the Greek “penta” meaning “five” and “kubos” meaning “cube”.

Two examples of pentacubes:



1. Build as many pentacubes as possible out of the unit cubes. Can you find all 29 different pentacubes?

2. In order to keep track of the large number of pentacubes, it makes sense to classify them into groups. Think of categories and name them. In your exercise book, write down what system you used and the names of your groups. Give a brief description of each of your groups.

3. Get the worksheet *Pentacubes Overview* from your teacher. Check if you have found all pentacubes. What system is used to order the pentacubes on the worksheet? Describe this system in your exercise book.

4. Examine the pentacubes for symmetry.

5. Find solutions for the following tasks concerning pentacubes and write them down in your exercise book.

   a) How many cubes do you need to create a full set of pentacubes (consisting of all 29 different pentacubes)?

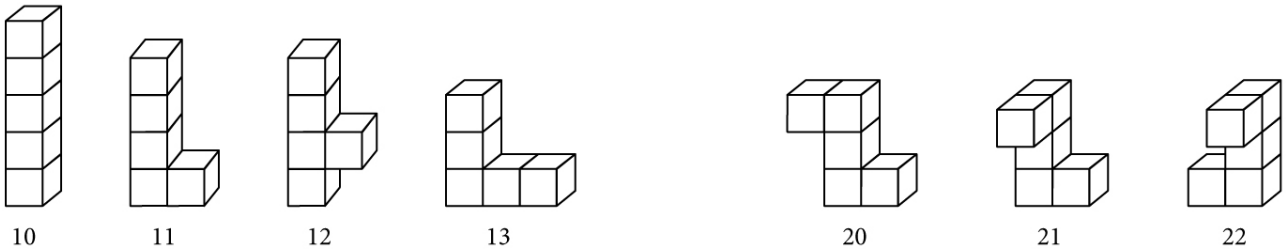
   b) How often do you glue two cubes together when you build all 29 different pentacubes?

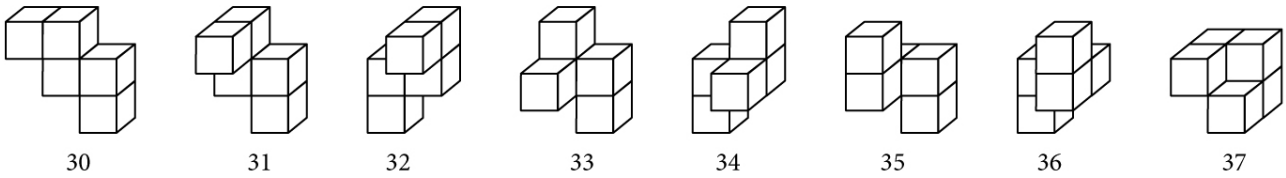
   c) Ann would like to paint her set of pentacubes. For which ones does she need the most (the least) paint? Why?

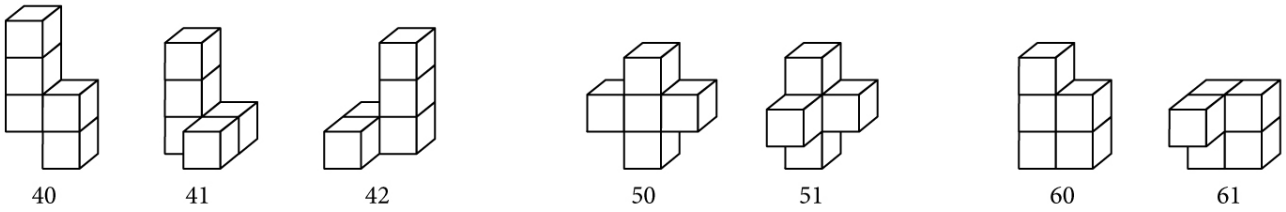
Name: Date:

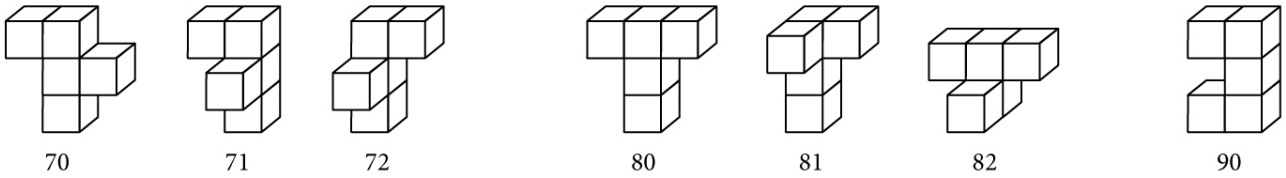
Pentacubes Overview

You can find all possible pentacubes here:







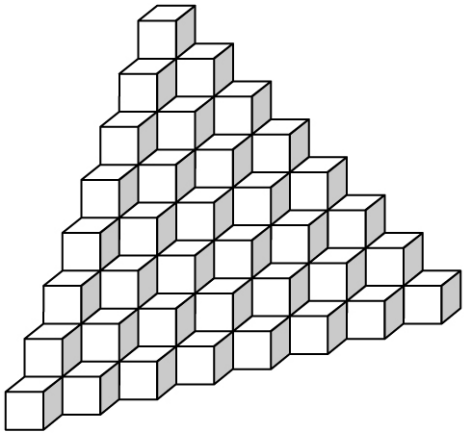


Name: Date:

Building with Pentacubes

1. Using pentacubes, you can create different buildings.
   The building to the right is one impressive example.

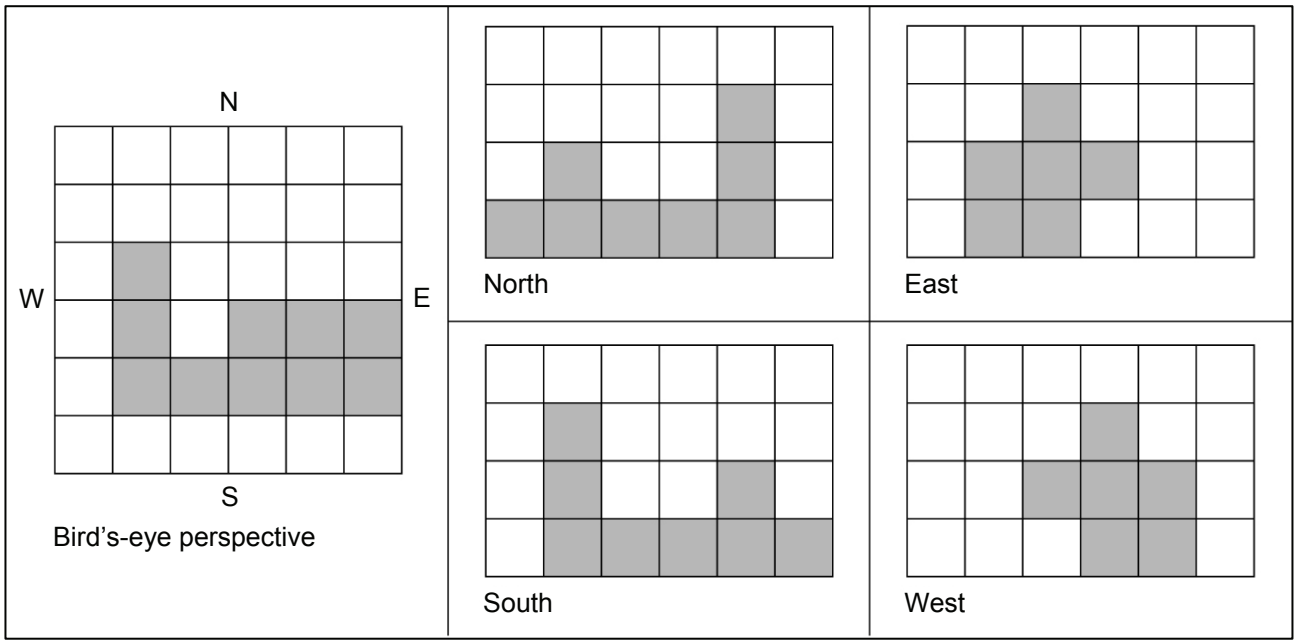
   Create such buildings, big ones, small ones, and try to avoid holes in your buildings.



2. Now you can create buildings out of pentacubes based on a plan. You see the side views of the building form North, East, South and West and the bird's-eye view. Additionally, you are told which pentacubes you need (numbered like on the worksheet *Pentacubes Overview*)

Building 1

Building 1 consists of the Pentacubes 10, 50 and 82.

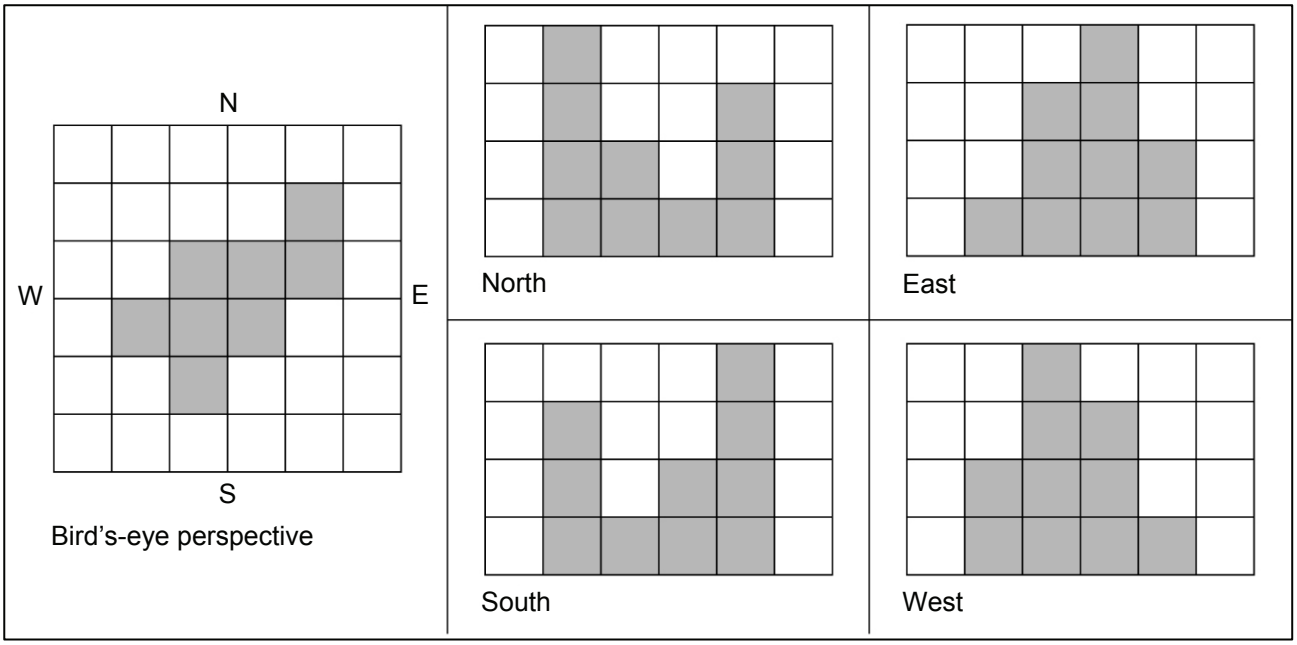


Name: Date:

Building with Pentacubes

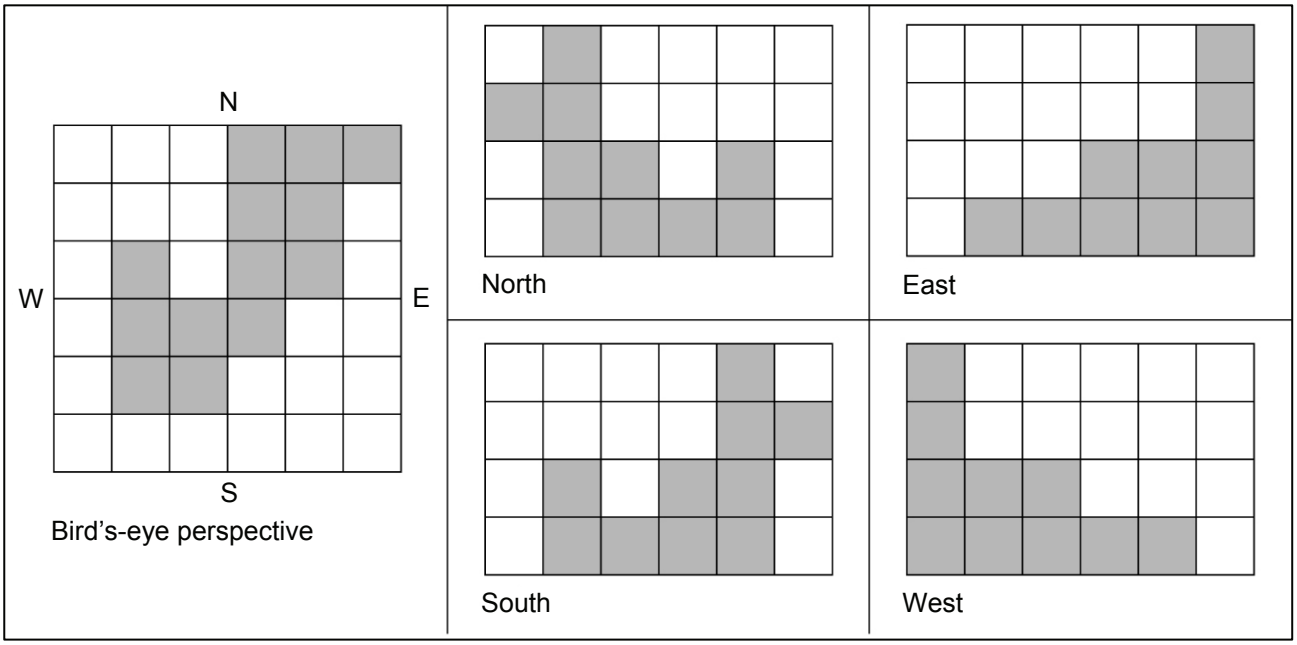
Building 2

Building 2 consists of the pentacubes 40, 41 and 51.



Building 3

Building 3 consists of the pentacubes 21, 31, 37 and 70.

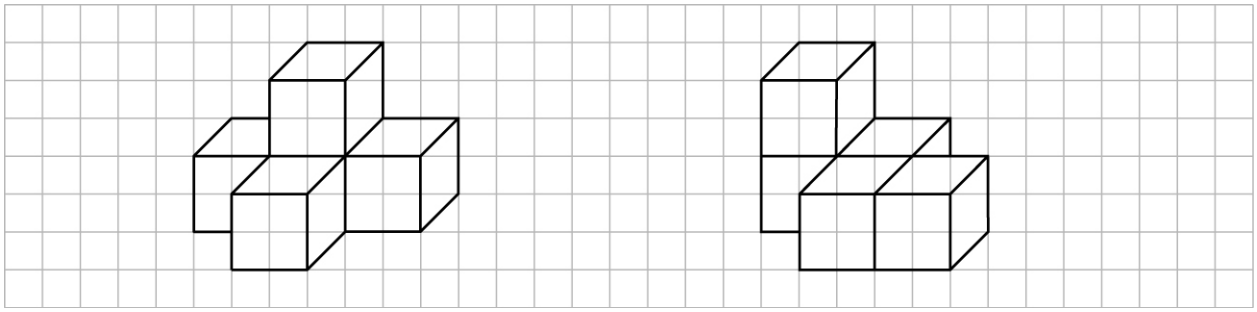


Name: Date:

Drawing Pentacubes: Making Two Dimensions Out of Three

It is not easy to present a three dimensional object on paper. Nevertheless, you can make oblique drawings on graph paper.

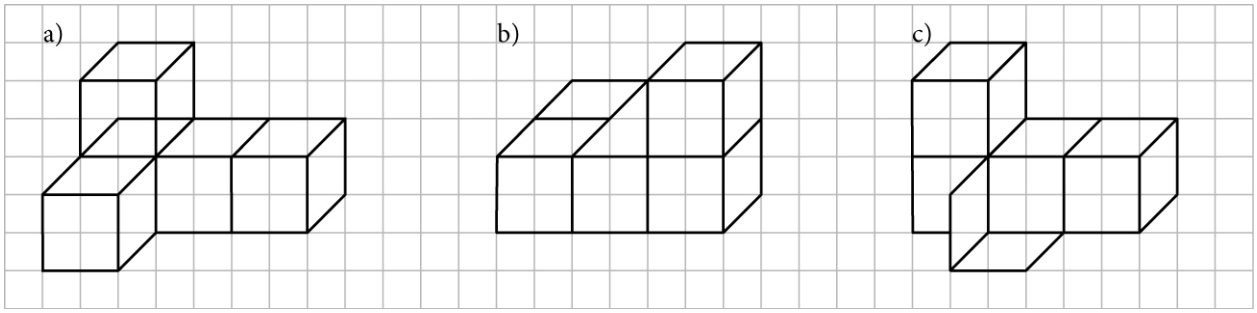
1. Which pentacubes do you see here? Write the number from the *Pentacubes Overview* worksheet next to the drawing.



2. Explain how to make an oblique drawing on graph paper.

3. Choose a pentacube and make oblique drawings in your exercise book. Check the drawings with your neighbour.

4. There are some mistakes in the following oblique drawings. Correct the mistakes with red color, and describe the mistakes in your exercise book.



5. Make oblique drawings with mistakes in your exercise book. Your neighbour should find and correct the mistakes.

Name: Date:

How to Play Pentacube Quartet

Preparation
Pentacube Quartet is played with three or more players. You need 32 playing cards that show the pentacubes.

Four cards show the same pentacube, but from different perspectives. This set of four cards is one quartet.

The cards are shuffled and dealt evenly between all the players. It is possible that some players get one more card more than the others.

Start
Each player checks to see if he or she already has a quartet. If he or she has one, the cards creating the quartet are placed in front of the player. The other players make sure that there are not any wrong cards in the quartet. If there is a wrong card, the player will have to take back his or her cards, and the player who found the mistake may draw two cards from the player who made the error.

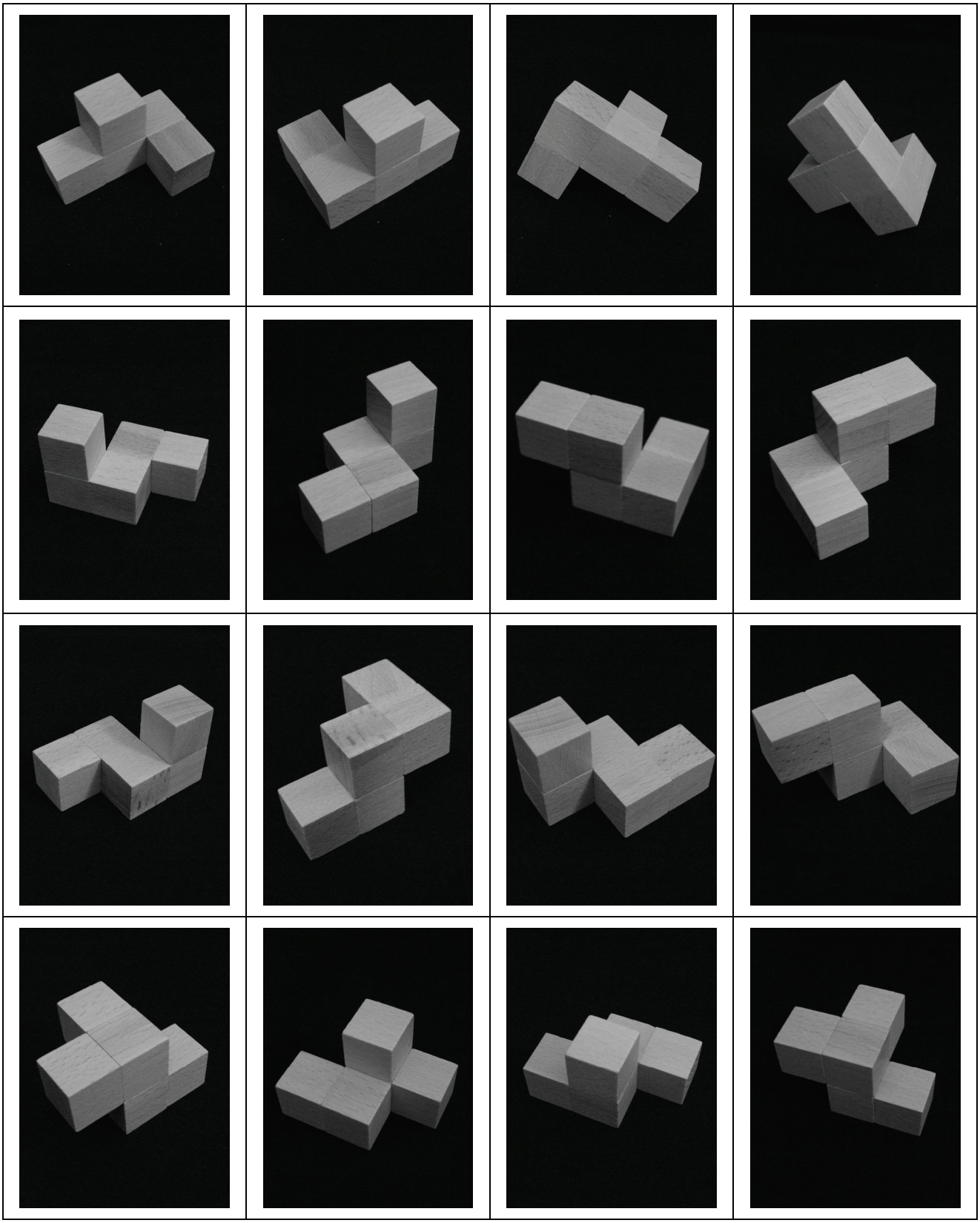
When no more quartets can be displayed, the youngest player starts and draws a card from his or her neighbour on the left. When a quartet is created, it is placed in front of the player, and the other players control the quartet as mentioned above.

Gameplay
One after another, the players draw a card from his or her neighbour on the left. When a quartet is created, it is placed in front of the player. The other players control the quartet as mentioned above.

The winner is the person with the most quartets.

Name: Date:

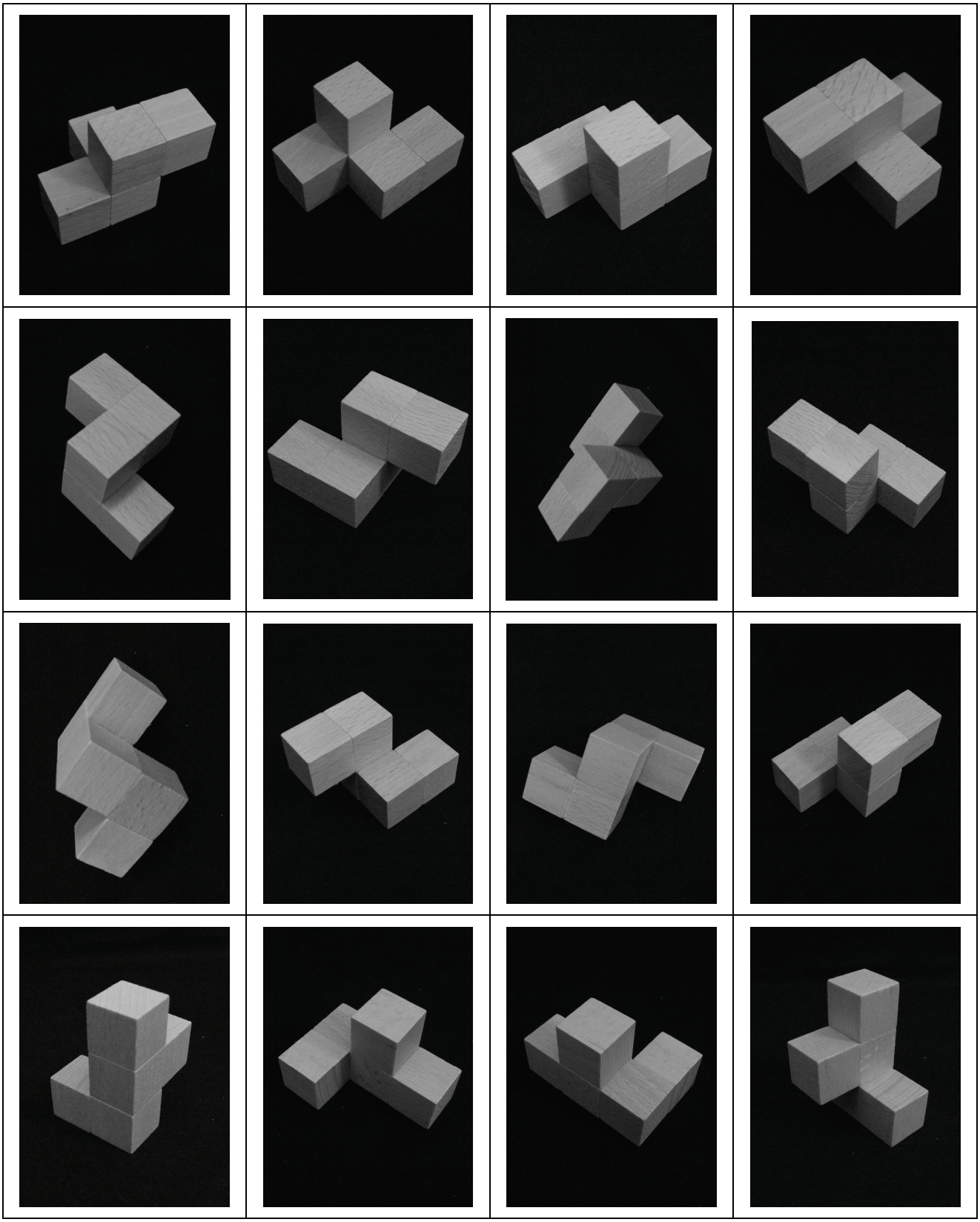
Pentacube Quartet: Playing Cards (1)



Photos: Birgit Brandl

Name: Date:

Pentacube Quartet: Playing Cards (2)



Photos: Birgit Brandl

14 Measuring Our School
Deepening the Idea of Length

T. Sinning, A. Stix-Pöhner

Topic and Purpose

This chapter offers pupils the chance to get to know the different kinds of length units and the principle of measurement. Children always want to explore their environment. Therefore, it is challening for the pupils to have a closer look at their school building and its rooms, which are part of their everyday life. Thus, the process of measuring makes them more aware of their environment.

Background

Children in the early years of primary school have intuitive and unsystematic ideas concerning length. These lessons should build on these ideas. The pupils get the opportunity to grasp realistic ideas about the range of length measurements and to learn benchmarks (such as a big step or an arm span is approximately the same as one meter). Within the process of abstraction, the pupils learn about how long certain familiar objects are. Therefore, they will soon be able to estimate measurements of unknown objects.

Methodological Advice

1. Using Body Measurements, a Ruler and Measuring Tape

The children get to know the body measurements and measure different objects at their places or in the classroom. Then they should compare their results with those of other pupils and explain what they have noticed. It should be mentioned that their results will differ due to the different sizes of their body parts. As a result, the children understand the necessity of having standardized units of measurement. The teacher might give some background information about the international treaty in Paris in 1875, when representatives of 17 states agreed to standardize units of measurement, particularly the standard meter and the standard kilogram.

Afterwards, the pupils should learn and practice how to measure exactly by using a ruler and a measuring tape. It is recommended to let the children first guess, then let them measure exactly. Thus, they can check any former inaccurate ideas of length they may have had. By doing several exercises, they will soon have a better idea of measurements.

2. A Plan of Our Classroom

The pupils should cut out the different objects such as desks, shelves, etc. and make a plan of their classroom. They can glue the objects on a sheet of A3 paper and write down the measurements of the previously measured objects on this plan. Other objects from their classroom can be added. Of course, the pupils can also draw the plan by themselves without cutting out the objects printed on the worksheet. In the end, the pupils should reflect their solutions in class.

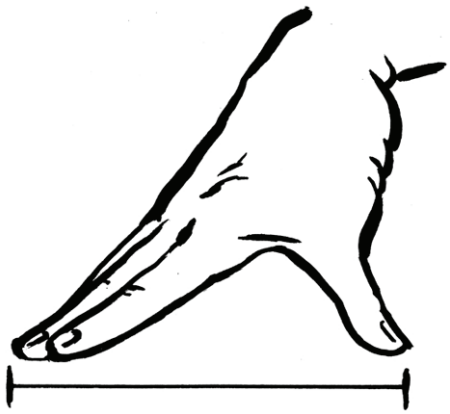
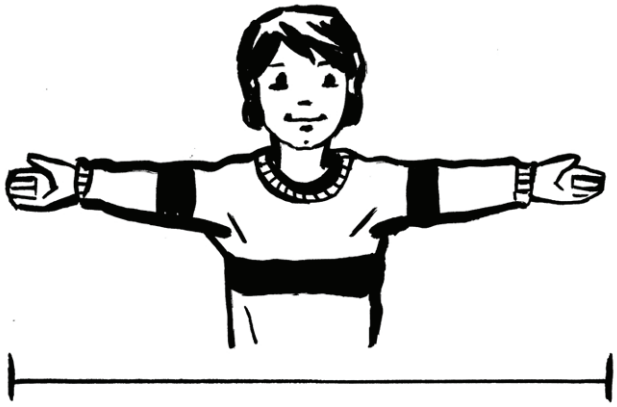
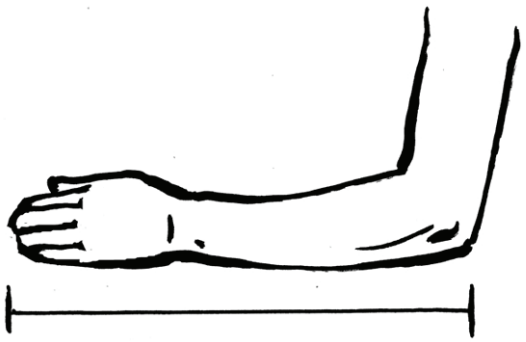
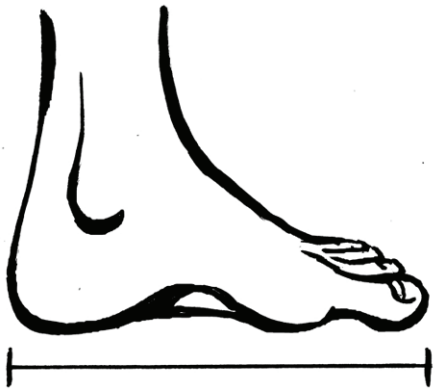
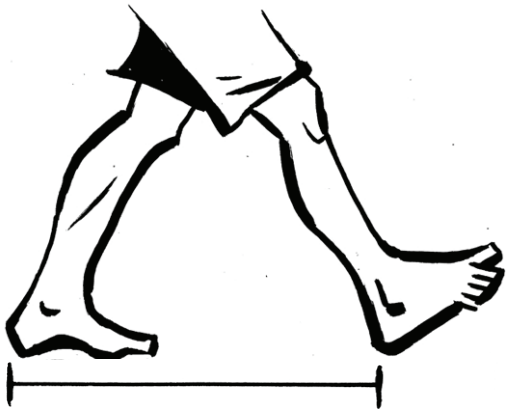
3. Our School from Outside

The pupils should estimate the height and width of their own school building. Former exercises concerning estimating and measuring help them to do so. They go outside, look at their school building from outside and count windows and floors, name classrooms, etc. After having made a first draft including distinctive objects such as windows, doors or the roof, they estimate the height and the width of the school building. Then, they measure certain lengths. By lowering a long measuring tape out of a window, they can measure the height of the first floor. Thus, they can create their own units of measure (height of a floor, classroom width, etc.). Combining these units will help the pupils to find out the lengths that are difficult to access, such as the height of the whole school building.

Name: Date:

Our Body Measurements

Fill in the terms for each type of body measurement:
foot, thumb, step, forearm, hand span, arm span

Name: Date:

Using Body Measurements

Use body measurements for measuring objects.

At your place

	hand span	thumb	forearm		
desk width					
desk length					
desk height					
chair height					
your height					

In the classroom

	forearm	arm span	foot	step	
room width					
room length					
window width					
window height					
black board width					
door height					

Compare your results with those of your partner or of your group.
What do you notice? Write down your observations and explain.

Name: Date:

Using a Ruler and a Measuring Tape

Measure objects with a ruler or measuring tape. First, make a guess for each object.

At your place

	my guess	measured
desk width		
desk length		
desk height		
chair height		
your height		

In the classroom

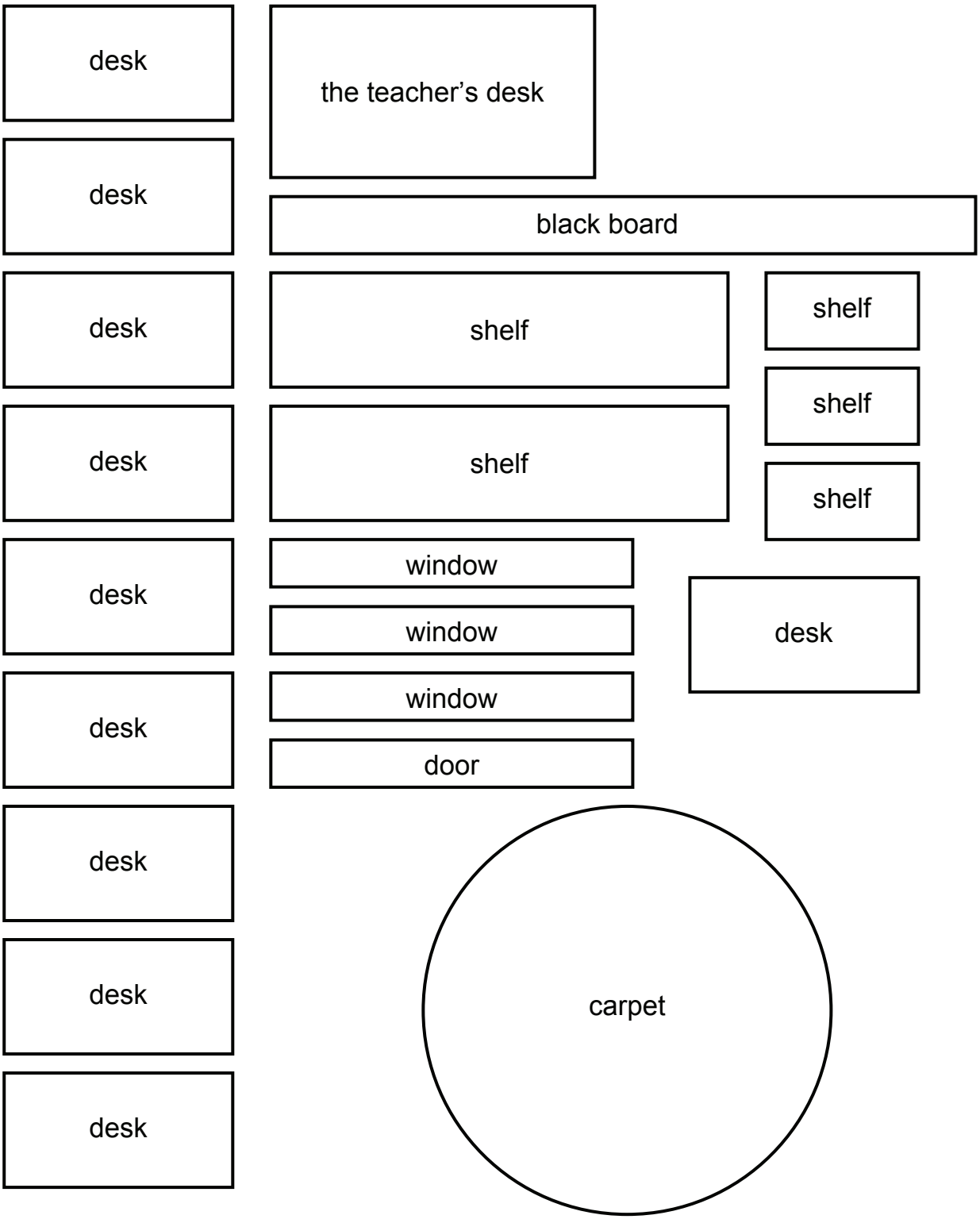
	my guess	measured
room width		
room length		
window width		
window height		
black board width		
door height		

Compare your results with those of your partner or of your group.
What do you notice? Exchange your ideas and explain your observations.

Name: Date:

A Plan of Our Classroom

Make a plan of your classroom. Cut out the objects and glue them on a big piece of paper. You can add more objects from your classroom on the plan. Write the measurements or dimension of each object on the plan.



Name: Date:

Our School from Outside

1. Go outside so that you can see one whole side of your school building.

2. Guess how wide and tall your school building is and how big its windows, doors, and roofs are.

3. Make a drawing of your school building from outside. Try to draw all the windows, doors, roofs, etc.

4. Think of which lengths of your school building you can measure and which lengths you can calculate by combining the lengths you have already measured.

5. Measure lengths of your school building, so that you get as many lengths as possible by combining. Write your results down on your drawing.

6. Describe how you did it and how you found your results.

15 Fencing In a Sheep Pasture

Discovering Perimeters and Surface Areas of Polygons

K. Weigl

Topic and Purpose

The following geometry section is about the connection between the perimeters and the surface areas of different polygons. The children can practice and extend their knowledge of geometric shapes. While doing so, they can discover how surface areas and perimeters are linked. On a basic level, they find out that shapes with same perimeters can have different surface areas. On an advanced level, children can find different surface areas for given perimeters and afterwards compare, change and adapt the problems and solutions according to their level of mathematical abilities.

These geometric topics are embedded in the simple situation of fencing in a pasture. Hence, the pupils can connect calculating operations with a concrete situation and the illustration of the situation. The worksheet provides various possibilities for the pupils to make the exercises more complex themselves. They can change the original situation, and thus, they are challenged by new mathematical problems. There are hardly any limits.

Background

In geometry lessons pupils often get to know different shapes by their characteristics such as edges, sides and angles. The ratio of sides defines the exact shape of the surface. Furthermore, surface areas help to understand multiplication exercises (e.g. square meters, tiling rooms etc.). This section connects two of these aspects, namely shapes and surface areas. Thus, the pupils' previous knowledge will be deepened and increased. The exercises are embedded in an easily comprehensible context. By translating the real-life situation to mathematics and vice versa, the pupils are encouraged to talk about the exercises and their ideas. Moreover, carefully illustrating the situation is a basic and important skill the pupils should develop.

Methodological Advice

In order to provide adequate exercises for children who differ in their previous knowledge and mathematical abilities, there should be appropriate tactile and visual tools. For drawing, the children need squared paper and a ruler. Regarding the measurement, it is recommended to use the scaling ratio of the side of one square on the paper as one meter in reality. Then one square on the paper corresponds to one square meter in reality. The children do not need to know the unit m^2, yet, because they can indicate the surface area as the number of squares. However, the teacher might introduce the unit "square meter".

At least at the beginning, the pupils should only draw straight lines and right angles along the gridded squares of the paper. While working on this topic, the pupils can eventually be allowed to break from this guideline and use half of the squares or 45° angles.

Note that the number of the fence pieces is 50 and the number 50 is not divisible by 4. Therefore, the square is not a solution in the first exercises. After having worked on Exercise 1, the class should discuss the topic together in order to clarify the problem regarding the size of the pasture verbally.

The pupils do not need to work on the other exercises in given order. Since the children mostly continue working with different results from previous exercises resulting in different solutions, a correction can only be done by discussing the pupils' ideas.

Name: Date:

Fencing In a Sheep Pasture

Farmer Johnny would like to fence in a pasture for his 25 sheep. He has 50 pieces of fence, each which is 1 m long.

1. Compare his first two plans. What do you notice? Explain in detail!



2. Think about how farmer Johnny can fence in the pasture for his sheep and draw your own plans for him.
   (Use gridded paper, and let the side of 1 square on the paper correspond to 1 m in reality.)
   In which cases do the sheep have a lot of space, and in which cases do they have less space?

3. Farmer Johnny would like to fence in a pasture twice as large as the original pasture. How many more fence pieces should he buy?

4. At the edge of the pasture, there are 4 trees which farmer Johnny has to exclude from the pasture. For each tree, he needs 1 m of fence to go around the tree. Draw possible shapes of the pasture. In which cases do the sheep have a lot of space and in which cases do they have less space?

5. Farmer Johnny would like to divide his pasture into two parts which are connected by a path which is not more than 2 m wide. Find different solutions.
   In which cases do the sheep have a lot of space, and in which cases do they have less space?

6. Farmer Johnny has bought 10 more sheep. How could he change his pasture to provide for each sheep as much space as the sheep had before?

7. Create similar exercises and work on them.

16 Experimenting with Dice

Examining the Dot Patterns and Nets of Dice

D. Bocka

Topic and Purpose

Pupils know dice from everyday life. A normal dice consists of 6 equal-sized squares that have dot patterns on them. However, these patterns have a certain order such that the sum of the opposite numbers always equals 7.

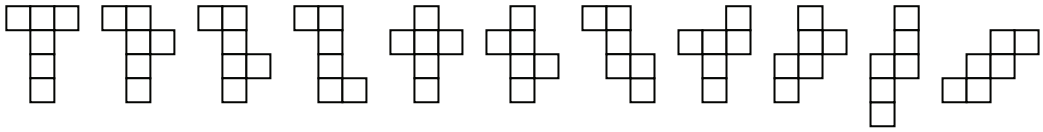
By working with these dot patterns, pupils from the 1st grade develop an understanding of quantities and their associated values. While playing, each dot pattern is automatically assigned to a numerical value. Examining these dot patterns in detail helps the children to recognize the different structures of a dice.

Further work is possible by investigating the nets of dice. Due to the fact that the sum of opposite sides of the dice is always the same, the pupils will also be trained in spatial-sense and the allocation of spatial-location.

Background

There is a major advantage of dot patterns in contrast to writing the numbers as numerals: you can easily read the value of each side regardless of the orientation, which is often necessary when playing a game. The usual writing on a dice is based on a 3x3 grid. Dot patterns up to 6 can be written in systematically.

There are eleven different nets of cubes. They consist of six equal and connected squares and can be classified by the longest row of squares. A row of four squares offers six different possibilities, such as the "T" and the "Cross". Using a row of three squares, there are four possibilities, and using a row of two squares, there is only one possibility for the net of a cube.



Methodological Advice

The worksheet *Examining Dice* can be used starting with the 1st grade as an introduction to the topic. The first part consists of an exercise on matching the amount of a number with the numeral. It is recommended to provide big dice for the pupils to look at.

While working on the worksheet *Inventing Dot Patterns*, it makes sense to discuss the suitability of the solutions that have been found. The teacher can mention that the patterns are unique, unmistakable and – even from different perspectives – easily read. Thinking about other dot patterns in everyday life is one possible continuation of the topic. For example, dot patterns can be found on the identification symbol for the blind or at traffic lights. These examples can introduce another topic such as figurate numbers (see the section within this book titled *Exploring the System*).

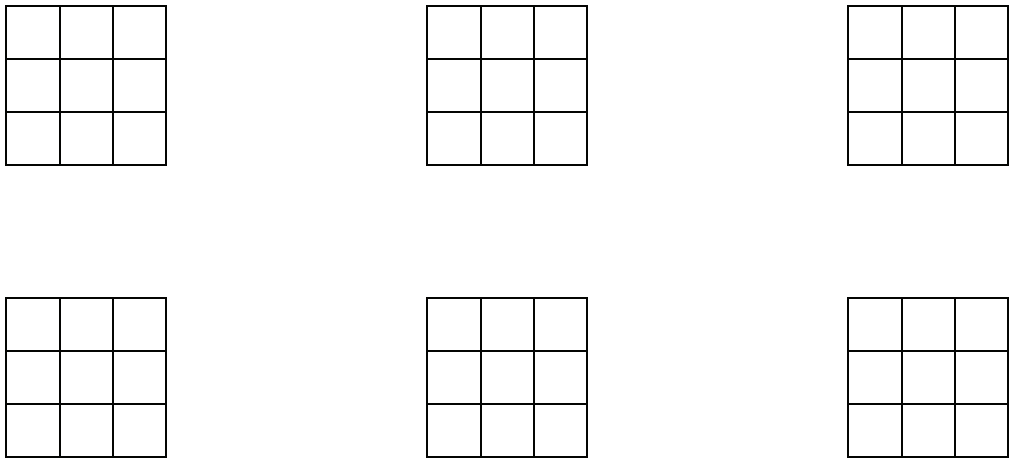
The solutions for the worksheet *Examining Nets of Dice* can be checked by cutting out and folding the cubes. Alternatively, a dice can be used for checking by "unrolling" its net. In order to further train the children's spatial-sense, it is important not only to focus on the standard nets of cubes.

As a continuation of this topic, the class can work on the nets of other three-dimensional shapes. The teacher can share unusual dice, which have different writing or shapes such as tetrahedron, octahedron, icosahedron or dodecahedron dice. These shapes and also the "normal" cubes belong to a group called the platonic solids.

Name: Date:

Examining Dice

1. Take a dice and look at its dots. Draw the dot patterns you find into the grids below and also write down the number of dots.



2. You can easily mix up the numerals 6 and 9, if they are upside down. By comparison, the dot patterns can be easily read from any orientation – even if they are upside down. Try this with the dice.

3. Find out which pairs of numbers are located on opposite sides to each other. What do you notice?

4. Add up all the numbers on the dice. What is the connection between this result and the result from exercise number 3?

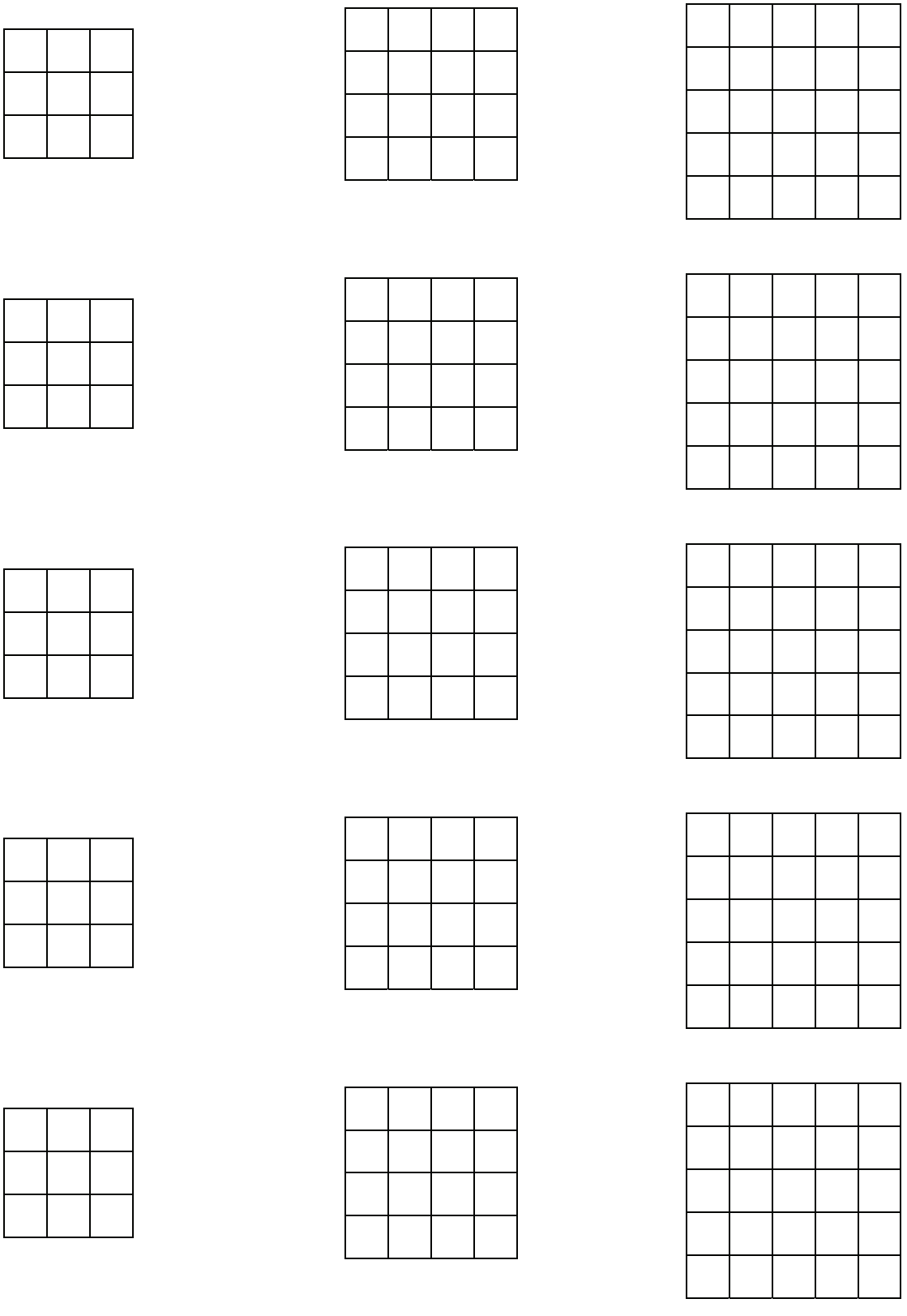
5. Put a dice in front of you on your desk. What is the sum of all the numbers that you can see?

6. Build a tower of two (or three, or four, …) dice. What is the sum of all the numbers you can see?

Name: Date:

Inventing Dot Patterns

Invent other dot patterns for the numbers 1 to 6, but also for bigger numbers. Write down the numbers next to the dot patterns. After completing this worksheet use your exercise book to continue. Which patterns are symmetric, and which are not?

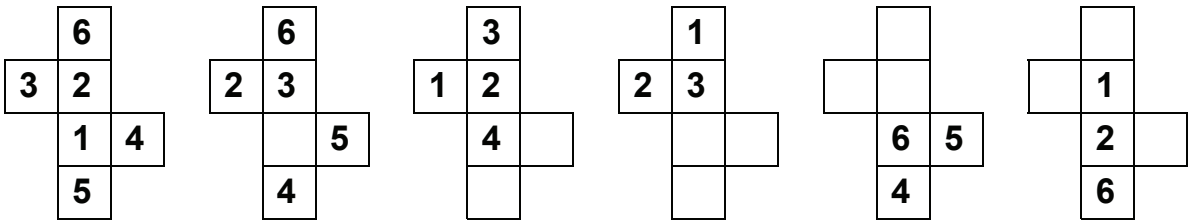


Name: Date:

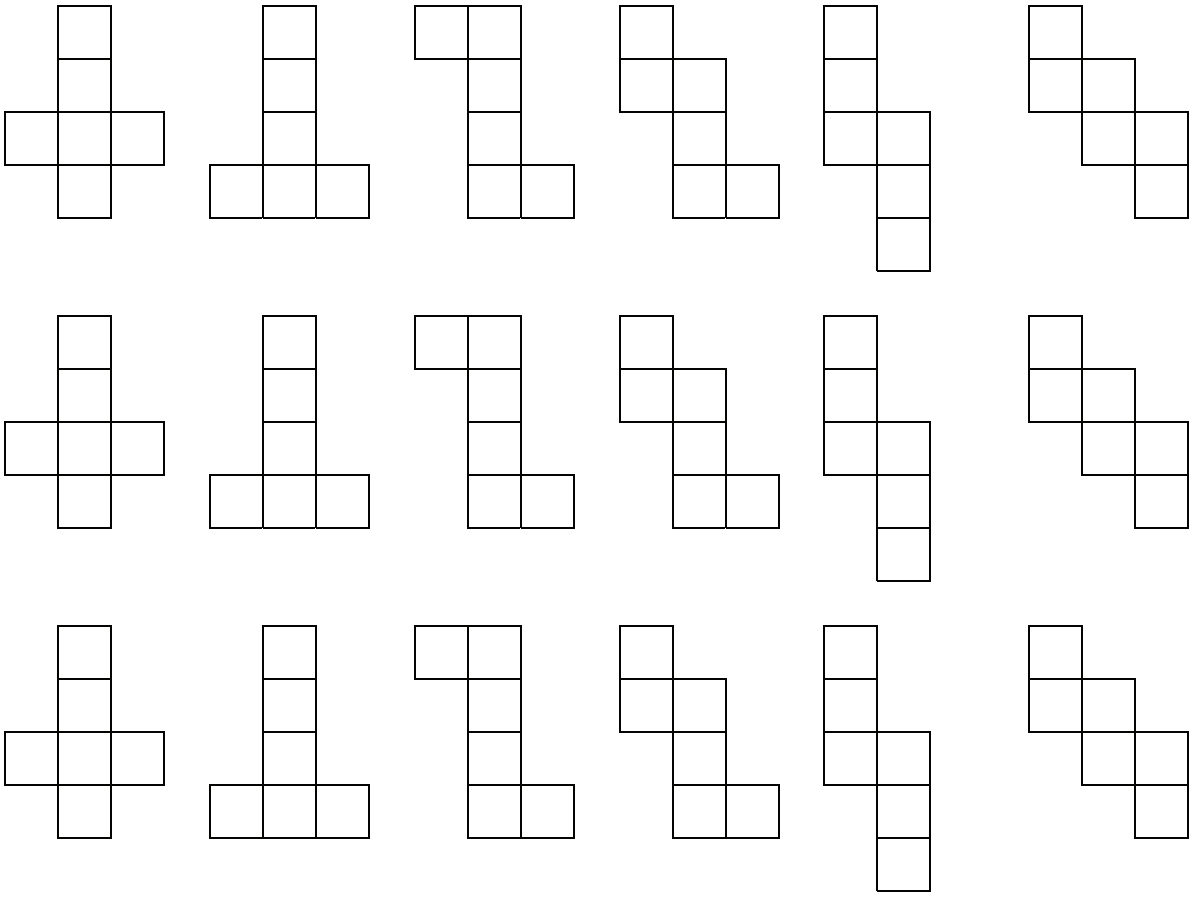
Examining Nets of Dice

1. Add up the numbers of the dice that are located on opposite sides to each other. What do you notice?

2. Now you can complete the missing numbers in the following nets of the dice below.



3. There are also other nets of dice. Write down possible numbers of a dice. Remember that the sum of the opposite numbers must always be the same.



17 Pentominoes

Geometric Shapes within a Plane

I. Weigand

Topic and Purpose

The exciting topic of pentominoes offers a lot for mathematical thinking and discovery (cf. Wittmann, Müller, 2007, p. 16). In particular, working with different shapes supports the pupils' geometrical thinking. The pupils recognize that shapes that seem to be different at first sight are the same after turning and flipping. These exercises are recommended for advanced pupils who work efficiently and quickly and like to be challenged.

Background

Pentominoes consist of five squares. These are placed so that every square is connected to another square by at least one whole side.

There are twelve different pentominoes all together. You can have an overview by putting them into the following order (see the worksheet): "Start with the simplest pattern. Change only one square to fit the next pattern." There is more than one solution; an example of one solution is found on the following page as a pupil's solution.

By playing the calendar game, the children will discover that it is possible to cover the calendar with pentominoes without overlapping such that you can only see a single day. This can be done for all 31 days. You can play the calendar game in two ways: You can either play it with 12 pentominoes or only with 7 pentominoes, the latter of which is more difficult (cf. Beutelspacher, 2008). For covering the calendar you need 6 pentominoes, because every pentomino covers five calendar squares.

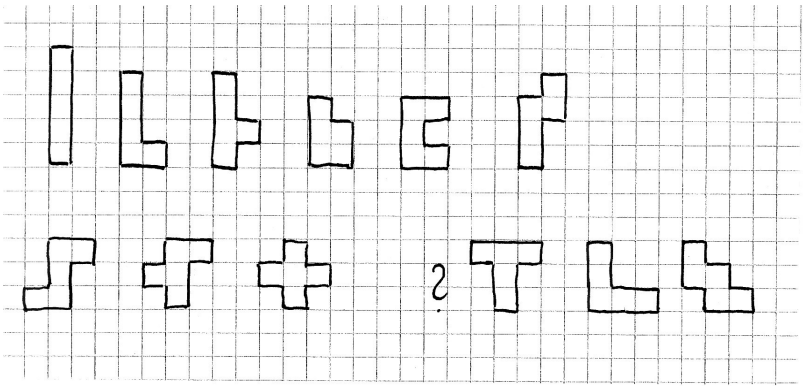
Methodological Advice

The pupils should approach the first activity *Pentominoes* by trial and error and working with these different patterns. For doing this, the pupils can use small wooden squares. Using this kind of material also helps the children to draw the pentominoes.

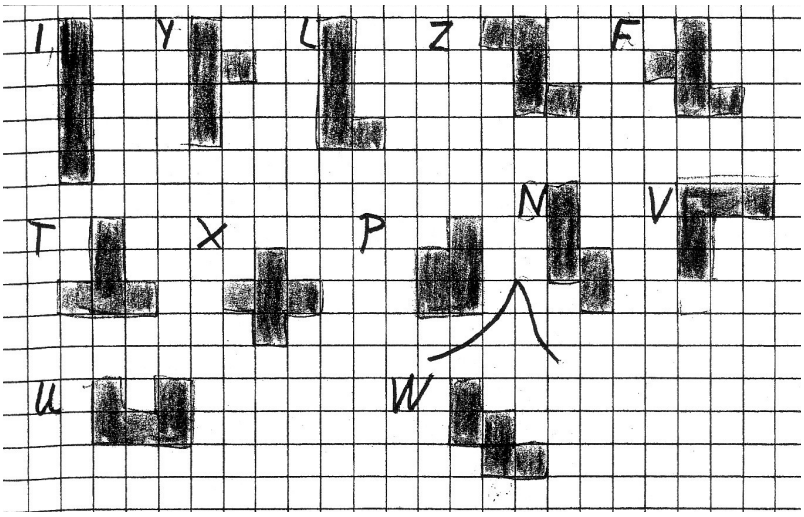
In order to play the calendar game, the pupils can cut out the pentominoes for themselves. The teacher could enlarge the worksheet to A3 and also laminate the pentominoes. Be sure that all single squares within the pentominoes are visible, as this helps to see the structure of the fields. All solutions that the pupils have found can be collected on a poster.

As an exercise throughout the whole year, the children can play the calendar game every day.

Pupils' drawings

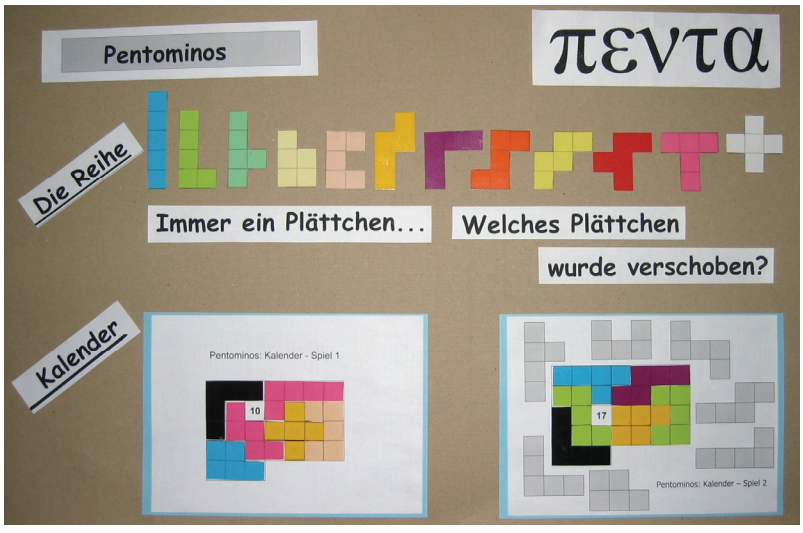


When creating a sequence of pentominoes, one or several pentominoes can be left over.



Here the pupil also forgot one pentomino at the end. He, who had already turned and flipped the pentominoes, briefly reflected on his solution for a second time and then quickly put the remaining figure in the right place.

Poster



This poster presents the different games. It is important for the pupils to have a closer look at the sequence of pentominoes. They should examine how one square changes from one picture to the next.

Further Reading

Beutelspacher, A., Wagner, M. (2008): Wie man durch eine Postkarte steigt, Herder, Freiburg

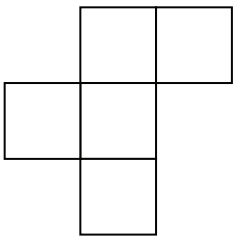
Wittmann, E., Müller, G. (1999): Spielen und Überlegen, Die Denkschule, Teil 2, Klett, Stuttgart

Wittmann, E., Müller, G. (2007): Das Zahlenbuch 3, Bayern, Klett, Stuttgart

Name: Date:

Pentominoes

1. Take five small squares and create different shapes. Every square should be connected to another square by at least one whole side. For example like this:



These figures are called pentominoes. This name comes from the Greek word “penta” meaning “five”.

2. Draw all the different pentominoes you can find on a piece of paper and cut them out. Each side of a square should be 2 cm long.

3. How many different pentominoes are there? Explain and give reasons.

4. Place the pentominoes into order according to the following rule:

   - Start with the simplest pentomino.
   - Change only one square to form the next shape.
   - At the end there should be no pentominoes left over.

   Draw this sequence. Can you find different sequences?

Name: Date:

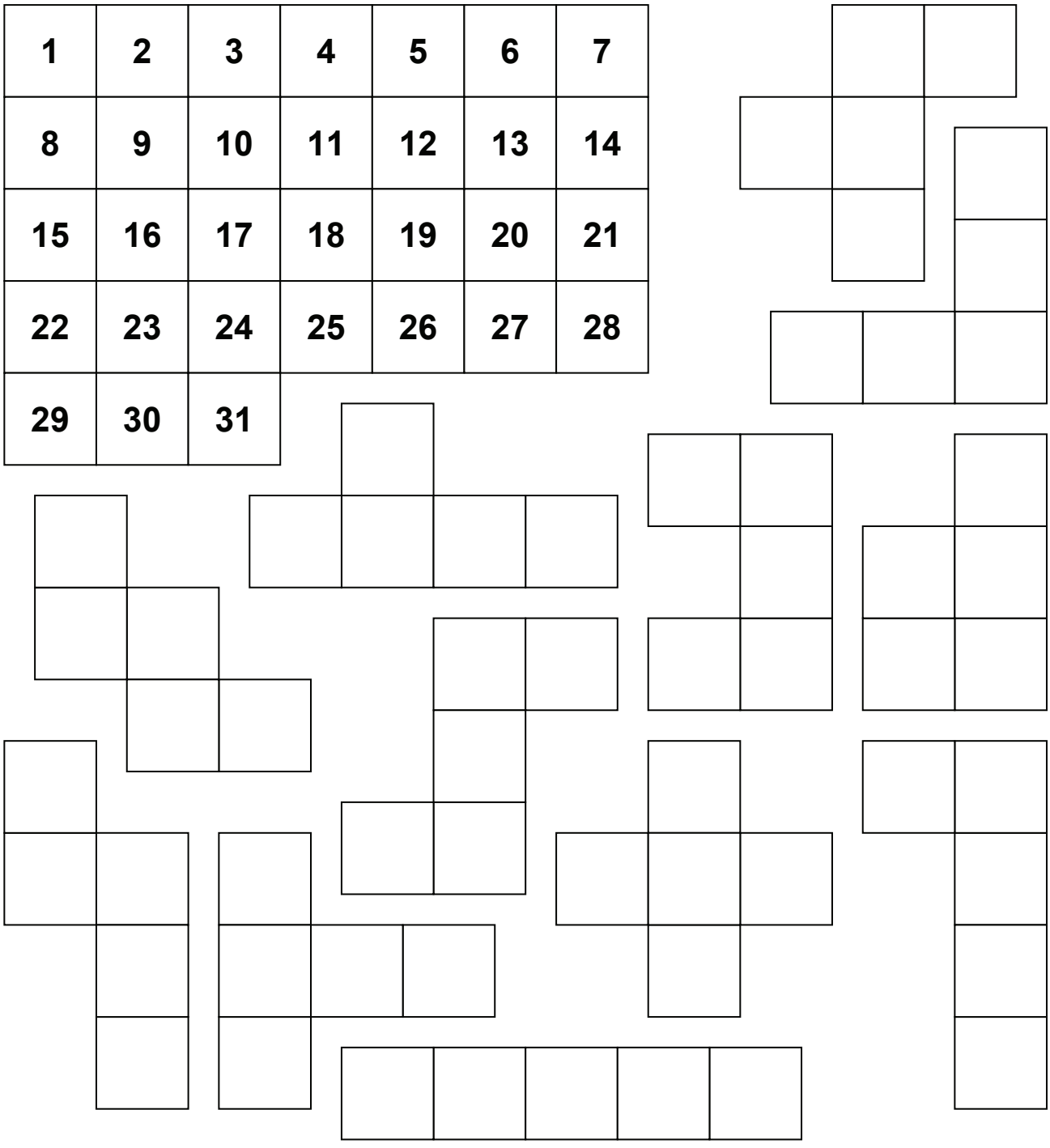
Pentominoes: The Calendar Game

1. Cut out all twelve pentominoes.

2. Put the pentominoes without overlapping on the calendar, so that you can only see a single day.

3. Try this for every day on the calendar. How many pentominoes do you have to use for each different day? Explain.

1	**2**	**3**	**4**	**5**	**6**	**7**
8	**9**	**10**	**11**	**12**	**13**	**14**
15	**16**	**17**	**18**	**19**	**20**	**21**
22	**23**	**24**	**25**	**26**	**27**	**28**
29	**30**	**31**				



Name: Date:

Pentominoes: A Challenging Calendar Game

1. Cut out all seven pentominoes.

2. Without overlapping, put the pentominoes on the calendar, so that you can only see one single day.

3. Try this for every day on the calendar. How many pentominoes do you have to use for each different day? Explain.

1	**2**	**3**	**4**	**5**	**6**	**7**
8	**9**	**10**	**11**	**12**	**13**	**14**
15	**16**	**17**	**18**	**19**	**20**	**21**
22	**23**	**24**	**25**	**26**	**27**	**28**
29	**30**	**31**				

